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ANNOUNCEMENT

INTERNATIONAL CONGRESS OF MATHEMATICIANS, 1954.

First Communication.

The International Congress of Mathematicians 1954 will be held in Amsterdam from September 2nd to September 9th under the auspices of "Het Wiskundig Genootschap" (The Mathematical Society of the Netherlands). It is the sincere hope of the "Wiskundig Genootschap" that the Congress 1954, which will be open to all mathematicians from all parts of the world, will be a fertile international gathering.

The Organizing Committee has invited a number of outstanding mathematicians to deliver one-hour addresses, hoping that in this way a survey of the recent development in the whole field of mathematics may be furnished.

There will be seven sections. *viz.*

1. Algebra and Theory of Numbers.
2. Analysis.
3. Geometry and Topology.
4. Probability and Statistics.
5. Mathematical Physics and Applied Mathematics.
6. Logic and Foundations.
7. Philosophy, History and Education.

Those who wish to attend the Congress are requested to communicate their name (with degrees, qualifications etc.) and full address to the secretariat as soon as possible. They will receive a more detailed communication in due course.

Amsterdam,
2d Boerhaavestraat 49.

The Organizing Committee.

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THERMODYNAMICS OF MOIST AIR

By

S. K. Banerji*, *Calcutta.*

Thermodynamics of the moist air, apart from its intrinsic interest as a mathematical problem, is of fundamental importance in the Physics of the Lower Atmosphere. In the development of this subject, work carried out in India has made significant contribution. Knoche (1905) and Hann (1907) were among the earliest writers on the subject. Knocke's object was to study the distribution of the heat content of the surface layers using equivalent temperature as a measure of the heat content; Hann's work contained some errors. In 1910, Neuhoff gave the adiabatic diagram for the atmosphere and this was a very valuable contribution. The most important contribution was made by Normand (1921). He gave rigorous definitions of equivalent temperature and equivalent potential temperature and obtained simple relations connecting them with the wet-bulb temperature. He deduced that the wet-bulb temperature and the equivalent temperature are each measure of the heat content of the air at constant pressure and further showed that the wet-bulb potential temperature and the equivalent potential temperature are invariants in all adiabatic processes. Since then much work has been done in this subject, the most important of which are those of Robitzch (1928), Rossby (1932), Stuve (1937), Bleeker (1939). Rossby considered equivalent temperature and equivalent potential temperature in detail. He defined equivalent potential temperature as the ultimate potential temperature assumed by air when all its water has been removed. This value of the equivalent potential temperature is approached asymptotically as the pressure in the element tends towards zero. In reviewing this subject, Brunt remarked that the use of equivalent potential temperature has not yet been thoroughly justified in practice. Most of the writers use a number of approximations and no analysis of the subject can be regarded as satisfactory unless the accuracy of the approximations is justified. Nagabhusana Rao has examined the causes for the differences in the equivalent potential temperature derived by using (*a*) Rossby's tables and (*b*) adiabatic diagrams. He has also given a more rigorous theoretical treatment of the subject and explanation of the various differences.

If we focus attention to a parcel of moist air anywhere in the atmosphere, it has in any particular position, a definite pressure, dry bulb-temperature and wet-bulb temperature. It continually changes its position, going up or down, during which all the above properties are changing. Further more in some stages condensation or evaporation might occur; there will also be radiation effects. The problem is—can we make a thermodynamical analysis and get a consistent picture of all the possible changes? The meteorologists have wanted that we should discover some property of the parcel of air which remains as a conservative property during all possible changes so that they could identify the parcel during its movements. The wet-bulb potential temperature and the equivalent potential temperature have certain amount of conservative character but no complete answer to the question which will hold in all circumstances has yet been discovered.

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* Address delivered at the Annual General Meeting of the Calcutta Mathematical Society held on January 29, 1954.

Entropy of ordinary air at standard pressure.

Let S_1 represent the entropy of dry air,
S_2, the entropy of water vapour,
S_3, the entropy of water,
S_x, the entropy of x gms. of water,
L, the latent heat of water-vapour at temperature T.

Then since the specific heat of water is approximately unity,

$$S_3 = \int^T \frac{dT}{T} = \log T + \text{const.}, \tag{1}$$

T being the absolute temperature of the dry-bulb thermometer.

Further,

$$S_2 = S_3 + \frac{L}{T}. \tag{2}$$

The entropy S of a mixture of 1 gm. of air and x gms. of water vapour is, therefore,

$$S = S_1 + xS_2 = C_p \log T + \frac{xL}{T} + S_x + \text{const.}, \tag{3}$$

where C_p is the specific heat of air at constant pressure.

If now the temperature and humidity are altered by dT and dx respectively, then

$$dS = C_p \frac{dT}{T} + \frac{Ldx}{T} + \frac{x}{T}\frac{dL}{dT}dT - \frac{xLdT}{T^2} + dS_x.$$

As a first approximation we can neglect the terms $-\frac{xLdT}{T^2}$ and $\frac{x}{T}\frac{dL}{dT}dT$, because x is of the order of 0·01; at ordinary atmospheric temperatures, these terms are therefore small compared with $C_p \frac{dT}{T}$. The equation, therefore, takes the form

$$dS = C_p \frac{dT}{T} + \frac{Ldx}{T} + dS_x. \tag{4}$$

If now the change has been such that the wet-bulb temperature of the air has been kept constant, then

$$C_p dT + Ldx = 0. \tag{5}$$

Therefore (4) gives

$$dS = dS_x.$$

This means that the entropy of the air mixture has been increased by the same amount as the entropy of the water has diminished, for dx grams of water must have been evaporated. In other words, the total entropy of the air mixture and water has been unaltered. This proposition which was originally due to Normand is true within the limits of approximation specified above.

Defining the *equivalent temperature*, T_E, as the temperature of absolutely dry air whose wet-bulb temperature is T', we get

$$C_pT_E = C_pT' + L'x', \tag{6}$$

where L' is the latent heat of water vapour at temperature T' and x' is the humidity mixing ratio of air saturated at the wet-bulb temperature T'. Using the approximate form of wet-bulb equation,

$$C_p(T-T') = L'(x'-x),$$

we can write

$$C_pT_E = C_pT' + L'x' = C_pL + L'x$$

or,

$$C_pdT_E = C_pdT + L'dx = C_pdT + Ldx, \text{ approximately.} \tag{7}$$

Entropy of ordinary air at any pressure.

Now, for any change in the system air, water vapour, water,

$$dQ = dE + dW, \tag{8}$$

by the first law of thermodynamics, where dQ represents the heat added, dE, the increase in internal energy and dW, work done. But

$$dE = JC_vdT + JL'dx, \tag{9}$$

where J is the mechanical equivalent of a unit of heat and C_v the specific heat of air at constant volume. Also

$$dW = Pdv + J(L-L')dx, \tag{10}$$

where P denotes pressure and v specific volume. Therefore

$$dQ = JC_vdT + JLdx + Pdv. \tag{11}$$

But the gas law, $Pv = RT$, gives

$$Pdv + vdP = RdT = J(C_p - C_v)dT \tag{12}$$

$$\therefore \quad dQ = JC_pdT + JLdx - vdP. \tag{13}$$

If h represents height

$$-vdP = dh \tag{14}$$

(7), (13) and (14), therefore give

$$dQ = JC_pdT_E + dh = JC_pd\left(T_E + \frac{h}{JC_p}\right). \tag{15}$$

If now we define the *equivalent potential temperature* as

$$\theta_E = T_E + \frac{h}{JC_p}, \tag{16}$$

we get

$$dQ = JC_pd\theta_E. \tag{15a}$$

For any adiabatic change, whether the process involves condensation of water vapour or not, $dQ = 0$.

Therefore, (15) gives

$$\frac{dT_E}{dh} = -\frac{1}{JC_p}, \tag{17}$$

which is the well-known adiabatic lapse rate for dry air.

Furthermore, for any adiabatic process, whether the process involves condensation or not, (16) gives

$$d\theta_E = 0, \tag{18}$$

or θ_E = constant. Therefore, within the limits of approximation involved in these deductions, we can say that for any adiabatic change, the equivalent potential temperature is an invariant. It will be observed that there is a considerable amount of approximation involved in this deduction.

Now, if as before S represents the entropy of $(1+x)$ gms. of moist air at temperature T, humidity mixing ratio, x, and at a height h, and if S_x is the entropy of x gms. of water at temperature T', then clearly $S-S_x$ must be equal to the entropy of 1 gm. of perfectly dry air at a height h and temperature T_E and therefore must also be to the entropy S_E of 1 gm. of perfectly dry air at sea-level and temperature θ_E.

Hence

$$S = S_E + S_x = C_p \log \theta_E + x \log T' + \text{const.} \tag{19}$$

Heat-content function of moist air.

The heat-content function H is given by

$$H = E + Pv.$$

In the system water, air, water-vapour, we can neglect the volume of water, its changes being negligible, compared to the gaseous phase

For 1 gm. of dry air

$$H_A = E_A + Pv = JC_vT + Pv \tag{20}$$

For 1 gm. of water vapour

$$H_W = E_W + JL \tag{21}$$

The heat content function for $(1+x)$ gram of moist air is, therefore,

$$H = H_A + xH_W = JC_vT + Pv + JL_x + E_Wx$$

But

$$Pv = RT = J(C_p - C_v)T.$$

Therefore

$$H = JC_pT + JLx + E_Wx = JC_pT_E + E_Wx$$

or

$$d(H - E_Wx) = JC_pdT_E \tag{22}$$

$d(H-E_W)$ is zero, if T_E is constant, no matter what the pressure is. Hence the heat content function H of $(1+x)$ gm. of moist air differs from that of 1 gm. of dry air at its equivalent temperature by the value of the internal energy of x gms. of water.

It is clear from (16) that the "equivalent potential temperature" θ_E is the temperature which a mass of air at the equivalent temperature T_E attains when it is brought adiabatically to tha standard pressure P_0 The "wet-bulb potential temperature," θ_W, is defined by the equation

$$\theta_E = \theta_W + \frac{Lx_0}{C_p}, \tag{23}$$

where x_0 is the humidity mixing ratio of saturated air at standard pressure and at temperature θ_0, θ_W is therefore a single-valued function of θ_E and *vice versa*. It is, therefore, obvious that the wet-bulb potential temperature is also an *invariant* for all adiabatic changes, within the limits of approximation, which has been specified above.

We shall now make an attempt to give a more rigorous treatment. The subscript α will refer to air, β to water vapour and γ to water.

Water Vapour.

We shall treat water vapour as a perfect gas, even when near or at saturation. In the atmosphere we seldom deal with saturated air above 30°*C*. At these low temperatures Callendar (*"Vaporisation" and "Thermodynamics," Encyclopaedia Brittanica*) states that the deviation from $pv = RT$ law is small. Clausius (*On the Mechanical Theory of Heat*) has given detailed discussion of the deviation of saturated vapour from $pv = RT$ law. He gives the following values for its specific weight (referred to air).

T	273°	323°	373°	423°	473°
Sp. Wt.	0·622	0·631	0·645	0·666	0·698

Now, if water vapour behaves as a perfect gas, then we should have

$$\text{Sp. Wt.} = \frac{M_\beta}{M_\alpha} = 0{\cdot}622,$$

M being the molecular weight

The assumption is, therefore, justified.

Thermodynamics of saturated vapour

Let L denote the latent heat and λ the internal latent heat. Then

$$L = \lambda + p_\beta(v_\beta - v_\gamma) \tag{24}$$

Consider 1 gm. of water, of which m gm. is saturated vapour and $(1-m)$ gm. is liquid. Then

$$v = mv_\beta + (1-m)v_\gamma. \tag{25}$$

Hence

$$dQ = Ldm + [C_\gamma(1-m) + C_s m]\, dT, \tag{26}$$

where is C_s the specific heat of saturated steam.

Therefore

$$dS = \frac{L}{T}\, dm + [\, C_\gamma(1-m) + C_s m]\, \frac{dT}{T}.$$

Now dS is a perfect differential. Therefore,

$$\frac{\partial}{\partial T}\left(\frac{L}{T}\right) = \frac{\partial}{\partial m}\left[\frac{1}{T}\left\{C_\gamma(1-m) + C_s m\right\}\right]$$

or

$$C_s - C_\gamma = T\,\frac{d}{dT}\left(\frac{L}{T}\right). \tag{27}$$

The second thermodynamic relation is

$$\left(\frac{\partial S}{\partial v}\right)_T = \left(\frac{\partial p}{\partial T}\right)_v$$

In the present case

$$\frac{\partial S}{\partial v}=\frac{\partial S}{\partial m}\cdot\frac{\partial m}{\partial v}=\frac{1}{v_\beta-v_\gamma}\,\frac{\partial S}{\partial m}=\frac{1}{v_\beta-v_\gamma}\cdot\frac{L}{T}$$

Hence

$$\frac{L}{T}=(v_\beta-v_\gamma)\frac{\partial S}{\partial v}=(v_\beta-v_\gamma)\frac{\partial p_\beta}{\partial T}\,. \tag{28}$$

This is the same as Clapeyron's equation. The subscript $\left(\frac{\partial p_\beta}{\partial T}\right)_v$ is omitted, because in this chage p is independent of v.

Neglecting v_γ compared with v_β, we get

$$\frac{L}{T}=RT\frac{d\log p_\beta}{dT}\,,$$

or

$$L=RT^2\frac{d\log p_\beta}{dT}\,. \tag{29}$$

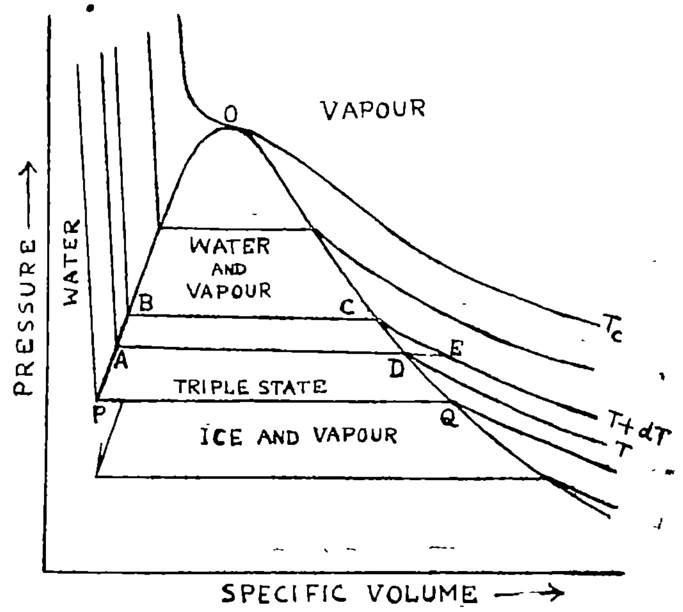

Fig. 1

In fig. 1, BC and AD are isothermals, along AB there is liquid state and along CD saturated vapour.

Therefore,
along AB, $DQ=C_\gamma dT$ (absorbed),
along CD, $dQ=-C_s dT$ (given out),
along DA, $dQ=-L$,
along BC, $dQ=L+\frac{dL}{dT}dT$,

Hence in the cycle $ABCDA$,

$$dQ=C_\gamma dT-L-C_s dT+L+\frac{dL}{dT}dT$$

$$=\left(C_\gamma-C_s+\frac{dL}{dT}\right)dT.$$

Now dQ must be equal to the external work done, that is, the area of the cycle

Therefore, $dQ=(v_\beta-v_\gamma)\left(\frac{\partial p}{\partial T}\right)_v dT$

Consequently

$$C_\gamma-C_s+\frac{dL}{dT}=(v_\beta-v_\gamma)\frac{dp_\beta}{dT}\,. \tag{30}$$

Further, the change of entropy in the cycle must be zero. Hence

$$\int_{cycl}\frac{dQ}{T}=C_\gamma\frac{dT}{T}+\frac{L+(dL/dT)dT}{T+dT}-C_s\frac{dT}{T}-\frac{L}{T}=0$$

or,

$$C_s - C_\gamma = \frac{dL}{dT} - \frac{L}{T}, \tag{31}$$

neglecting the term involving $(dT)^2$.

From (30) and (31), we get

$$\frac{L}{T} = (v_\beta - v_\gamma)\frac{dp_\beta}{dT}, \tag{32}$$

which is the same as (28).

From (24) and (32), we get

Internal latent heat

$$\lambda = T(v_\beta - v_\gamma)\left(\frac{dp_\beta}{dT} - \frac{p_\beta}{T}\right) = T^2(v_\beta - v_\gamma)\,\frac{d}{dT}\left(\frac{p}{T}\right).$$

Neglecting v_γ compared with v_β and using the relation

$$p_\beta v_\beta = RT$$

we get

$$\lambda = RT^2\frac{d}{dT}\left(\log\frac{p_\beta}{T}\right). \tag{33}$$

This form is similar but not the same as (29).

Taking the course DEC instead of DC in the cycle, we get

$$C_s = {}_\beta C_p + p\cdot\frac{\partial v_\beta}{\partial p_\beta}\cdot\frac{dp_\beta}{dT} \tag{34}$$

when ${}_\beta C_p$ is the specific heat of water vapour at constant pressure.

Now since

$$v_\beta = \frac{R_\beta T_\beta}{p_\beta},\quad \text{we get}$$

$$\frac{\partial v_\beta}{\partial p_\beta} = -\frac{R_\beta T_\beta}{p_\beta^{\,2}} = -\frac{v_\beta}{p_\beta}.$$

Therefore

$$C_s = {}_\beta C_p - v_\beta\frac{dp_\beta}{dT}. \tag{35}$$

From (30) and (35), we get

$$\frac{dL}{dT} = {}_\beta C_p - C_\gamma. \tag{36}$$

Also, from (24)

$$\frac{d\lambda}{dT} = \frac{dL}{dT} - p_\beta\frac{d(v_\beta - v_\gamma)}{dT} - (v_\beta - v_\gamma)\frac{dp_\beta}{dT} = C_s - C_\gamma - p_\beta\frac{d(v_\beta - v_\gamma)}{dT}, \tag{37}$$

using the relations (31) and (32).

Or,

$$\frac{d\lambda}{dT} = {}_\beta C_p - v_\beta\frac{dp_\beta}{dT} - p_\beta\frac{dv_\beta}{dT} - C_\gamma,$$

neglecting v_γ compared with v_β.

This can be expressed in the form

$$\frac{d\lambda}{dT} = {}_\beta C_p - \frac{d(pv_\beta)}{dT} - C_\gamma$$

$$= {}_\beta C_p - R - C_\gamma = {}_\beta C_v - C_\gamma. \tag{38}$$

From (32) and (35), we get

$$\frac{L}{T} = {}_\beta C_p - C_s. \tag{39}$$

Therefore

$$\frac{\lambda}{T} = \frac{L}{T} - R_\beta = {}_\beta C_v - C_s \tag{40}$$

Total heat (or the heat-content function, H), is given by

$$H = I + pv,$$

where I is the internal energy.

If the zero of the function is *water* at 0°C and at zero pressure, then the total heat of a saturated vapour is

$$H = C_\gamma(T-T_0) + L + p_\beta v_\gamma. \tag{41}$$

When $T-T_0$ is small, p_β is small and $p_\beta v_\gamma$ (expressed in heat units) is negligible. Therefore,

$$dH = C_\gamma dT + dL = {}_\beta C_p dT, \tag{42}$$

using (36). This would, however, not be true at high temperature.

Entropy of *saturated* vapour, taking the entropy of water at 0°C as zero, is

$$\left.\begin{aligned} S &= C_\gamma \log T + \frac{L}{T} - C_\gamma \log T_0 \\ &= {}_\beta C_v \log \frac{T}{T_0} + R_\beta \log \frac{v_\beta}{v_0} + \frac{L_0}{T_0} \\ &= {}_\beta C_p \log \frac{T}{T_0} - R_\beta \log \frac{p}{p_0} + \frac{L_0}{T_0} \end{aligned}\right\} \tag{43}$$

Here p_0, v_0 are saturated values at T_0, and p, v are saturated values at T.

Vapour Pressure Curves.

We can get the equation for the vapour pressure curve, using the method of derivation adopted by Callendar.

The fundamental relation is

$$T\frac{dp}{dT} = \frac{1}{v_\beta - v_\gamma}.$$

If we assume $p(v_\beta - v_\gamma) = R_\beta T$, that is, practically a perfect gas and also assume L constant, we get

$$\log \frac{p}{p_0} = \left(\frac{1}{T_0} - \frac{1}{T}\right)\frac{L}{R_\beta},$$

which is a formula of the type

$$\log \frac{p}{p_0} = A + \frac{B}{T}.$$

But L is not constant. If we assume low pressures and temperatures and also that C_γ and ${}_\beta C_p$ are practically constants, then

$$L = L_0 + ({}_\beta C_p - C_\gamma)(T - T_0).$$

We obtain

$$\left.\begin{aligned} \log p &= A+\frac{B}{T}+C\log T, \\ \text{where}\quad A &= \log p_0-\frac{B}{T_0}-C\log T_0 \\ B &= -[L_0-({}_\beta C_p-C_\gamma)T_0]/R_\beta \\ C &= ({}_\beta C_p-C_\gamma)/R \end{aligned}\right\} \qquad (44)$$

Entropy of unsaturated vapour.

In Fig. 2, T' is the dew-point of the unsaturated vapour at the point P.

Entropy of the unsaturated vapour at P can be expressed in the following forms.—

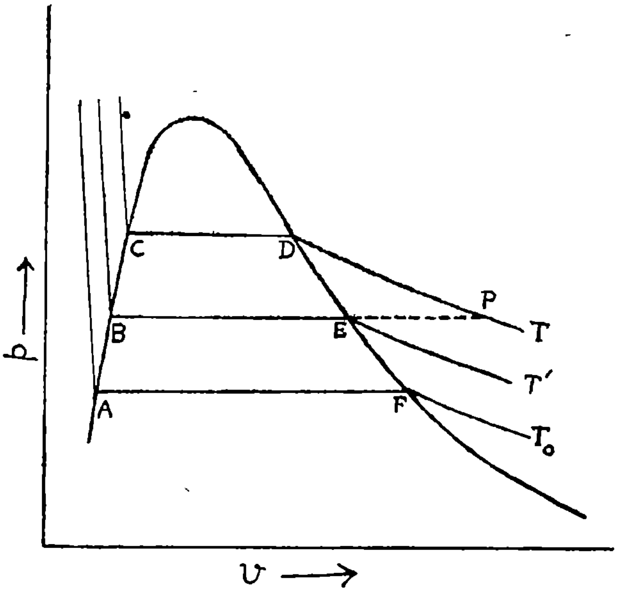

Fig. 2.

$$\left.\begin{aligned} (ACDP),\quad S &= C_\gamma\log\frac{T}{T_0}+\frac{L}{T}+R\log\frac{v}{v_s} \\ (ABEP),\quad S &= C_\gamma\log\frac{T'}{T_0}+\frac{L'}{T'}+C_p\log\frac{T}{T'} \\ (AFP),\quad S &= C_v\log\frac{T}{T_0}+\frac{L_0}{T_0}+R\log\frac{v}{v_0} \\ &= C_p\log\frac{T}{T_0}+\frac{L_0}{T_0}-R\log\frac{p}{p_0} \end{aligned}\right\} \qquad (45)$$

In general, the value of the entropy depends on the initial and final states only. Therefore, we can use

$$\left.\begin{aligned} S &= S_0 + C_\gamma \log \frac{T}{T_0} + \frac{L}{T} + R \log \frac{v}{v_s} \\ &= S_0 + C_\gamma \log \frac{T}{T_0} + \frac{L}{T} - R \log \frac{p}{p_s} \end{aligned}\right\} \quad (46)$$

Numerical Computations.

The following table, which is a repetition, with slight modifications of that given by Ewing ("*Steam Engine, Encyl. Brit.*) provides the basic data.

Properties of saturated water-vapour.

Temperature		Pressure,* (*mm. of Hg.*)	v cc. per gm.	Total Heat (H)		L	$\frac{L}{T}$	Entropy (S)	
T	$T-T_0$			Water	Vapour			Water	Vapour
273	0	4·579	$10^5 \times 2·049$	0	594·7	594·7	2·178	0	2·178
278	5	6·541	×1·470	5·0	597·1	592·1	2·130	0·018	2·148
283	10	9·205	×1·066	10·0	599·4	589·4	2·083	0·036	2·119
288	15	12·779	×·7823	15·0	601·8	586·8	2·037	0 054	2·091
293	20	17·510	×·5812	20·0	604·1	584·1	1·993	0·071	2·064
298	25	23·690	×·4367	25·0	606·5	581·5	1·951	0·088	2·039
303	30	31·710	× 3318	30·0	608·8	578·8	1·911	0·104	2.015
308	35	42·020	×·2540	35·0	611·1	576·1	1·870	0·121	1·991

* To convert to dynes/cm², multiply by $1·333 \times 10^3$.

In his article on "*Vaporisation*" (*Encyl. Brit.*), Callendar gave the following figures

$T-T_0$	H (vapour)	L	$_sC_p$
0°	595·2	595·2	0·4786
20°	604·7	584·7	0·4796
40°	614·0	574·0	0·4818

Now C_γ has the following values

$T-T_0$	0°	10°	20°	30°
C_γ	1·0088	1·0016	0·9989	0·9976

Therefore C_γ is approximately 1·00 (calories) at 15°C, and this can be taken as the mean value over the range of temperature.

Mean value of $$\frac{dL}{dT} = -\frac{594\cdot7-578\cdot8}{30} = -\cdot530.$$

Hence $${}_\beta C_p = C_\gamma + \frac{dL}{dT} = 0\cdot470.$$

Also, the mean value of

$$\frac{dH}{dT} = \frac{608\cdot8-594\cdot7}{30} = 0\cdot470 = {}_\beta C_p.$$

$$\therefore \quad L = 594\cdot7 - 0\cdot530\ (T-T_0)$$

Now R (molecular, all gases) = 1·985,

$$R_\beta = 0\cdot1103 \text{ cal. per } {}^\circ C$$

$$= 4\cdot62\times10^6 \text{ ergs per gm.}$$

$$\therefore \ {}_\beta C_v = {}_\beta C_p - R_\beta = 0.360$$

Hence $${}_\beta C_p / {}_\beta C_v = 1\cdot305$$

With these values, the vapour pressure curve (44) takes the form

$$\log_{10} p = 0\cdot6640 + 8\cdot585\ \frac{T-T_0}{T} - 4\cdot70\left(\log_{10}\frac{T}{T_0} - M\ \frac{T-T_0}{T}\right), \tag{47}$$

where $M = 0\cdot4343$ (log modulus).

The vapour pressure, for ice, reduces to the form

$$\log_{10} p = 0\cdot6640 + 9.73\ \frac{T-T_0}{T} \tag{48}$$

Entropy Calculation of a mixture of 1 gm. of air, x gm. of vapour and y gm. of water.

Example. Unsaturated air $T-T_0 = 35^\circ\cdot8$,

$$x = \frac{14}{1000} \text{ gms.,} \qquad \text{Height} = 0,$$

$$T = 308\cdot8, \quad P = 760 \text{ mm.} \quad P-p = 743\cdot4 \text{ mm.}$$

$$y = \frac{4\cdot7}{1000} \text{ gms. at } T-T_0,\ L = 581\cdot6$$

Now

$$S_{air} = C_p \log T - \frac{R}{J}\log(P-p) + \text{const.}$$

$$S_{vapour} = S_{water} + \frac{xL}{T}$$

$$S_{water} = {}_0S_w + \log\frac{T}{T_0}, \quad ({}_0S_w \text{ refers to } 0°C.)$$

Therefore, entropy of a mixture of 1 gm. of air $+ x$ gm. of vapour $+ y$ gm. of water is

$$S = (C_p + x + y)\frac{\log_{10} T}{M} + \frac{xL}{T} - \frac{R}{JM}\log_{10}(P-p) + \beta, \tag{49}$$

where $\beta = (x+y)\{{}_0S_w - \log T_0\} + \text{const.}$

$M = \text{modulus} = {\cdot}4343,$

$x + y = 0{\cdot}187, \quad C_p = 0{\cdot}2375,$

$C_p + x + y = 0{\cdot}2562, \qquad R/J = 0{\cdot}069.$

Therefore

$$\begin{aligned} S &= 1{\cdot}469 + 0{\cdot}0264 - 0{\cdot}456 + \beta \\ &= 1{\cdot}039 + \beta. \end{aligned}$$

Entropy-Temperature Diagram.

In the entropy-temparature diagram, given in Fig. 3, the isothermal lines are vertical lines and the dry-adiabats are horizontal lines, the pressure lines are the inclined lines, the moist-adiabats are curved lines which coincide with dry adiabats at low pressure. In a parcel of moist air, we can make the approximate statement that the wet bulb temperature follows the moist adiabat in the course of its movement. The moist adiabat thus traces its history. If we consider any particular moist adiabat, ABC, the equivalent potential temperature θ_E is the point, where the asymptote to the moist-adiabat intersects the standard pressure line (1000 mb.). The conservative property of θ_E for adiabatic processes is due to the fact that the asymptote identifies the moist adiabat. The wet-bulb potential temperature, θ_w, is determined by the point where the moist adiabat intersects the line of standard pressure. When the position of this point is known, the moist adiabat is completely identified. θ_w has therefore the conservative property for adiabatic processes.

Equal areas in this diagram represent equal amount of work or energy. The dotted lines represent the equi-moisture lines. In drawing the saturation adiabats, the equation usually adopted is an approximate form of (49) in which the entropy, S is made constant. The equation is expressed in the form

$$(C_p + \xi c)\frac{\log_{10} T}{M} + \frac{x_T L_T}{T} - A\,R\frac{\log_{10}(P-p)}{M} = \text{constant.}$$

The constants are usually taken to have the values given below.

C_p of dry air $= 0{\cdot}2400$ cal./gm, C_v of dry air $= 0{\cdot}1714$ cal./gm. $R = C_p - C_v = 2{\cdot}871 \times 10^6$ ergs/gm/°C.

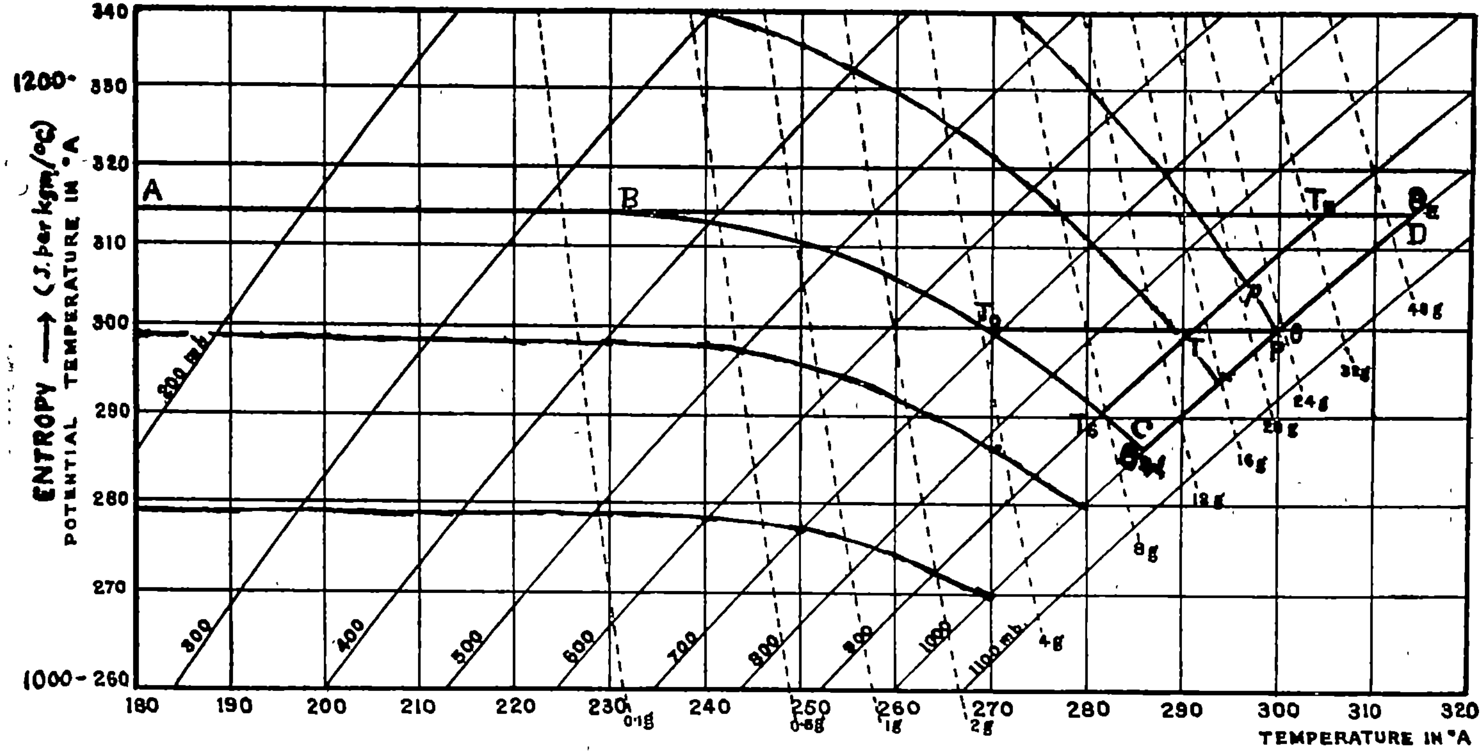
1200-
1000-
ENTROPY ⟶ (J. per kgm/°C)
POTENTIAL TEMPERATURE IN °A
340
330
320
310
300
290
280
270
260
180
190
200
210
220
230
240
250
260
270
280
290
300
310
320
TEMPERATURE IN °A
0·1g
0·5g
1g
2g
4g
8g
12g
16g
20g
24g
32g
40g
200 mb.
300
400
500
600
700
800
900
1000
1100 mb.
A
B
C
D
T
P
O
T_E
θ_E
T_D
T_S
θ_w

Fig. 3.

$J = 1/A = 4{\cdot}185 \times 10^7$ gm./cal., latent heat of vaporisation of water (L_T) $= 597{\cdot}3 - 0{\cdot}565(T-273)$ cal./gm., the latent heat of fusion of ice is 79.7 cal./gm, The value of ξ is taken as zero for all temperatures below 273°A. For higher temperatures varying values of ξ are used for the different saturation adiabats, the value for a particular adiabat being taken to be approximately equal to the mean of the mixing ratios at the point where it cuts the 1000 mb. isobar and the 273°A isotherm.

In view of the approximations adopted in thermodynamic analysis of moist air, many of the deductions made in actual practice become liable to considerable errors.

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ON CERTAIN FUNCTIONS WHICH ARE SELF-RECIPROCAL UNDER A NEW TRANSFORM

By

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1. Let

$$\varpi_{\mu,\nu}(xy) = \sqrt{(xy)} \int_0^\infty J_\nu(t) J_\mu(xy/t) t^{-1} dt; \quad -R(\mu+\tfrac{3}{2}) < 0 < R(\nu+\tfrac{3}{2}), \tag{1}$$

It has the following properties

$$(i) \quad \varpi_{\mu,\nu}(xy) = \varpi_{\nu,\mu}(xy).$$

$$(ii) \quad \varpi_{\nu,\nu-1}(xy) = J_{\nu-1}(2\sqrt{(xy)})$$

$$(iii) \quad \varpi_{\mu,\nu}(x) = O(1) \text{ when } x \text{ is small.}$$

$$(iv) \quad \varpi_{\mu,\nu}(x) = O(x^{-\frac{1}{2}}) \text{ when } x \text{ is large.}$$

Prof. Watson (1931) has shown that if

$$g(x) = \int_0^\infty \varpi_{\mu,\nu}(xy) f(y) dy, \tag{2}$$

then it might be possible to obtain the relation

$$f(x) = \int_0^\infty \varpi_{\mu,\nu}(xy) g(y) dy. \tag{3}$$

In this short note we give examples of several functions which satisfy (2) and (3) when $g(x)$ is replaced by $f(x)$ and hence are self-reciprocal.

Also we shall investigate the (formal) conditions under which the self-reciprocal property holds good.

2. (*a*) Let $f(y) = K_n(y) y^{n+\mu+\frac{1}{2}}$; $\nu = 2n+\mu$, $n \geq 0$.

Then

$$x^{\frac{1}{2}} \int_0^\infty \int_0^\infty K_n(y) J_{2n+\mu}(t) J_\mu(xy/t) y^{n+\mu+1} t^{-1} dt\, dy$$

$$= x^{\frac{1}{2}} \int_0^\infty J_{2n+\mu}(t) t^{-1} dt \int_0^\infty K_n(y) J_\mu(xy/t) y^{n+\mu+1} dy$$

$$= 2^{n+\mu} \Gamma(n+\mu+1) x^{\mu+\frac{1}{2}} \int_0^\infty \frac{J_{2n+\mu}(t) t^{2n+\mu+1} dt}{(t^2+x^2)^{n+\mu+1}}$$

$$= K_n(x) x^{n+\mu+\frac{1}{2}}, \quad -1 < R(2n+\mu) < R(2n+2\mu+\tfrac{3}{2}), \quad n \geq 0. \tag{4}$$

showing that $K_n(x)x^{n+\mu+\frac{1}{2}}$ is self-reciprocal under this transform, when $\nu = 2n+\mu$ and $n \geq 0$.

(*b*) Let $x) = x^{\frac{1}{2}}J_0(x/\sqrt{2})K_0(x/\sqrt{2})$.

We know that

$$\int_0^\infty \frac{xJ_0(ax)dx}{(x^4+1)^{1/2}} = J_0(a/\sqrt{2})K_0(a/\sqrt{2})$$

By Hankel's Theorem [Titchmarsh (1937), p. 240]

$$f_1(\varrho) = \int_0^\infty \int_0^\infty xyJ_n(x\varrho)J_n(xy)f_1(y)dxdy, \quad n > -\tfrac{1}{2}$$

provided that $f_1(x)x^{\frac{1}{2}}$ is continuous and absolutely integrable in $(0, \infty)$.

Hence

$$\int_0^\infty xJ_0(x\varrho)J_0(xk)K_0(xk)dx = 1/(\varrho^4+4k^4)^{\frac{1}{2}}$$

Therefore

$$x^{\frac{1}{2}}\int_0^\infty \int_0^\infty y^{\frac{1}{2}}J_0(t)\ J_0(xy/t)\ f(y)\ \frac{dt}{t}\ dy$$

$$= x^{\frac{1}{2}}\int_0^\infty J_0(t)\ \frac{dt}{t}\int_0^\infty yJ_0(xy/t)J_0(y/\sqrt{2})k_0(y/\sqrt{2})\,dy$$

$$= x^{\frac{1}{2}}\int_0^\infty \frac{J_0(t)tdt}{(x^4+t^4)^{\frac{1}{2}}} = x^{\frac{1}{2}}J_0(x/\sqrt{2})K_0(x/\sqrt{2})\,; \tag{5}$$

showing that $x^{\frac{1}{2}}J_0(x/\sqrt{2})K_0(x/\sqrt{2})$ is self-reciprocal in this transform.

(*c*) We know that

$$\int_0^\infty \frac{x^{-\nu}J_\nu(ax)dx}{(x^2+k^2)^{\nu\pm\frac{1}{2}}} = \left(\frac{2a}{k^2}\right)\ \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)}\ I_\nu(\tfrac{1}{2}ak)K_\nu(\tfrac{1}{2}ak), \quad R(\nu) > -\tfrac{1}{2}.$$

By Hankel's Theorem, we get

$$\int_0^\infty x^{3/4}J_{-\frac{1}{4}}(x\varrho)I_{-\frac{1}{4}}(\tfrac{1}{2}x)K_{-\frac{1}{4}}(\tfrac{1}{2}x)dx = \frac{2^{\frac{1}{4}}\Gamma(\frac{1}{2})\varrho^{-3/4}}{\Gamma(\frac{3}{4})(\varrho^2+\ )^{\frac{1}{4}}}.$$

Let

$$f(y) = y^{\frac{1}{4}}I_{-\frac{1}{4}}(\tfrac{1}{2}y)K_{-\frac{1}{4}}(\tfrac{1}{2}y) \text{ and } \mu = \nu = -\tfrac{1}{4}.$$

Then

$$\int_0^\infty \int_0^\infty (xy)^{\frac{1}{4}}J_{-\frac{1}{4}}(t)J_{-\frac{1}{4}}\left(\frac{xy}{t}\right)f(y)\,\frac{dt}{t}\ dy$$

$$= \int_0^\infty x^{\frac{1}{4}}J_{-\frac{1}{4}}(t\ \frac{dt}{t}\int_0^\infty y^{3/4}J_{-\frac{1}{4}}\left(\frac{xy}{t}\right)I_{-\frac{1}{4}}(\tfrac{1}{2}y\,K_{-\frac{1}{4}}(\tfrac{1}{2}y)dy$$

$$= \frac{2^{\frac{1}{4}}\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}\ \int_0^\infty \frac{x^{-\frac{1}{4}}t^{\frac{1}{4}}}{(x^2+t^2)^{\frac{1}{4}}}\ J_{-\frac{1}{4}}(t)\ dt$$

$$= x^{\frac{1}{4}}\ I_{-\frac{1}{4}}(\tfrac{1}{2}x)\ K_{-\frac{1}{4}}(\tfrac{1}{2}x)$$

(*d*) Let $f(y) = (\pi/2)(I_0(y) - L_0(y))$, $\mu = \nu = \frac{1}{2}$

In Hankel's theorem, let us put $n = \frac{1}{2}$ and take $f_1(x) = 1/\sqrt{\{x(x^2+1)\}}$.

After a little simplification, we get

$$1/\sqrt{(\varrho^2+1)} = \int_0^\infty (I_0(x) - L_0(x)) \sin x\varrho \, dx$$

Therefore

$$\int_0^\infty \varpi_{\frac{1}{2}, \frac{1}{2}}(xy) f(y)\, dy$$

$$= x^{\frac{1}{2}} \int_0^\infty \sin t \frac{dt}{t} \int_0^\infty \sin \frac{xy}{t} (I_0(y) - L_0(y))\, dy$$

$$= \int_0^\infty \frac{\sin t\, dt}{\sqrt{(x^2+t^2)}}$$

$$= \tfrac{1}{2}\pi(I_0(x) - L_0(x));$$

showing that $\frac{1}{2}\pi\,(I_0(x) - L_0(x))$ is self-reciprocal when $\mu = \nu = \frac{1}{2}$.

(*e*) We have the integral*

$$I = \int_0^\infty \frac{x^{m+1} J_{m+2}(ax)\, dx}{(x^2+K^2)^{m+1}} = \frac{(-1)^{m+1}}{2^m m!} \left(\frac{d}{KdK}\right)^m \{K^m K_{m+2}(aK)\} = F(K) \text{ say}$$

By Hankel's theorem, we have

$$\frac{\varrho^m}{(1+\varrho^2)^{m+1}} = \int_0^\infty \frac{(-1)^{m+1} x^{m+1} J_{m+2}(x\varrho)}{2^m\, m!} \left(\frac{d}{xdx}\right)^m \{x^m K_{m+2}(x)\} dx$$

Let

$$f(y) = \frac{(-1)^{m+1}}{2^m m!} y^{m+\frac{1}{2}} \left(\frac{d}{ydy}\right)^m \{y^m K_{m+2}(y)\},\ \mu = \nu = m+2,\ m = a \text{ positive integer}$$

Then

$$\frac{(-1)^{m+1} x^{\frac{1}{2}}}{2^m m!} \int_0^\infty J_{m+2}(t) \frac{dt}{t} \int_0^\infty J_{m+2}(xy/t)\, y^{m+1} \left(\frac{d}{ydy}\right)^m \{y^m K_{m+2}(y)\}\, dy$$

$$= x^{m+\frac{1}{2}} \int_0^\infty \frac{t^{m+1} J_{m+2}(t)\, dt}{(x^2+t^2)^{m+1}} = \frac{(-1)^{m+1}}{2^m m!} x^{m+\frac{1}{2}} \left(\frac{d}{xdx}\right)^m \{x^m K_{m+2}(x)\} = F(x)$$

(*f*) Let

$$F(y) = \int_0^\infty \sinh^{\frac{1}{2}} t\, K_0(y \sinh t)\, dt$$

$$= \int_0^\infty \frac{K_0(yp)\, p^{\frac{1}{2}}\, dp}{\sqrt{(p^2+1)}}$$

---

*The integral is convergent and we can put $I = F(K)$ and proceed as usual with $F(K)$ as the value of the integral.

Also let $f(y) = y^{\frac{1}{2}} F(y),\ \mu = \nu = 0$

$$\int_0^\infty \int_0^\infty (xy)^{\frac{1}{2}} J_0(t) J_0\left(\frac{xy}{t}\right) f(y) \frac{dt}{t}\, dy$$

$$= x^{\frac{1}{2}} \int_0^\infty \frac{J_0(t)}{t} dt \left\{ \int_0^\infty \int_0^\infty \frac{p^{\frac{1}{2}} y J_0(xy/t) K_0(yp)}{\sqrt{(p^2+1)}}\, dp\, dy \right\}$$

$$= x^{\frac{1}{2}} \int_0^\infty \frac{J_0(t)}{t} dt \left\{ \int_0^\infty \frac{p^{\frac{1}{2}}\, dp}{\sqrt{(p^2+1)}} \int_0^\infty y J_0\left(\frac{xy}{t}\right) K_0(yp)\, dy \right\}$$

$$= x^{\frac{1}{2}} \int_0^\infty J_0(t)\, t\, dt \int_0^\infty \frac{p^{\frac{1}{2}}\, dp}{\sqrt{(p^2+1)}(x^2+p^2t^2)}$$

$$= x^{\frac{1}{2}} \int_0^\infty \frac{dp}{p^{3/2}\sqrt{(p^2+1)}} \int_0^\infty \frac{J_0(t)\, t\, dt}{(x^2/p^2)+t^2}$$

$$= x^{\frac{1}{2}} \int_0^\infty \frac{p^{\frac{1}{2}} K_0(xp)\, dp}{\sqrt{(p^2+1)}} = x^{\frac{1}{2}} F(x)$$

(*g*) Let $f(y) = \frac{1}{8}\pi^2 \left(J_0^2(\frac{1}{2}y) + y_0^2(\frac{1}{2}y)\right);\ \mu = \nu = -\frac{1}{2}$

$$= \int_0^\infty \frac{K_0(py)}{\sqrt{(p^2+1)}}\, dp$$

Therefore

$$\int_0^\infty \int_0^\infty (xy)^{\frac{1}{2}} J_{-\frac{1}{2}}(t)\, J_{-\frac{1}{2}}(xy/t)\, f(y)\, \frac{dt}{t}\, dy$$

$$= \frac{2}{\pi} \int_0^\infty \int_0^\infty \cos t \cos\frac{xy}{t} f(y) \frac{dt}{t} dy$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos t}{t} dt \left\{ \int_0^\infty \cos\frac{xy}{t} dy \int_0^\infty \frac{K_0(py)}{\sqrt{(p^2+1)}} dp \right\}$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos t}{t} dt \left\{ \int_0^\infty \frac{dp}{\sqrt{(p^2+1)}} \int_0^\infty \cos\frac{xy}{t} K_0(py)\, dy \right\}$$

$$= \int_0^\infty \cos t\, dt \int_0^\infty \frac{dp}{\sqrt{(p^2+1)}\sqrt{(p^2t^2+x^2)}}$$

$$= \int_0^\infty \frac{dp}{p\sqrt{(p^2+1)}} \int_0^\infty \frac{\cos t\, dt}{\sqrt{(t^2+x^2/p^2)}} = \int_0^\infty \frac{K_0(px)\, dp}{\sqrt{(p^2+1)}}$$

$$= \frac{1}{8}\pi^2 \left(J_0^2(\tfrac{1}{2}y) + K_0^2(\tfrac{1}{2}y)\right) \tag{9}$$

(*h*) Let $f(y) = \frac{1}{4}\pi^2 y^{\frac{1}{2}}(H_0(y) - Y_0(y) - \pi/4y^{\frac{1}{2}};\ \mu = \nu = 0.$

We know that

$$\int_0^\infty \frac{K_0(py)\, dp}{p^2+1} = \frac{1}{4}\pi^2 (H_0(y) - Y_0(y))$$

Then

$$
\begin{aligned}
x^{\frac12}\int_0^\infty\int_0^\infty y^{\frac12} J_o(t)\, J_o(xy/t)\, f(y)\frac{dt}{t}dy
&= x^{\frac12}\int_0^\infty J_o(t)\, t^{-1}dt\left[\int_0^\infty J_o(xy/t)dy\int_0^\infty \frac{y\,K_o(py)}{p^2+1}\,dp-\tfrac12\pi\int_0^\infty J_o(xy/t)dy\right]\\
&= x^{\frac12}\int_0^\infty J_o(t)\, t^{-1}dt\left[\int_0^\infty \frac{t^2\,dp}{(p^2+1)(p^2t^2+x^2)}-\frac{\pi}{4}\,\frac{t}{x}\right]\\
&= x^{\frac12}\int_0^\infty\int_0^\infty \frac{t\,J_o(t)\,dt\,dp}{p^2(p^2+1)(t^2+x^2/p^2)}-\frac{\pi}{4x^{\frac12}}\int_0^\infty J_o(t)\,dt\\
&= x^{\frac12}\int_0^\infty K_o'(px)\,dp-x^{\frac12}\int_0^\infty\frac{K_o(px)}{p^2+1}\,dp-\frac{\pi}{4x^{\frac12}}\\
&= \frac{\pi}{4x^{\frac12}}-\frac{\pi^2}{4}x^{\frac12}(H_o(x)-Y_o(x))-\frac{\pi}{4x^{\frac12}}\\
&= \frac{\pi}{4x^{\frac12}}-\frac{\pi^2}{4}x^{\frac12}(H_o(x)-Y_o(x))\,; \qquad (10)
\end{aligned}
$$

showing that

$\frac{\pi^2}{4}y^{\frac12}(H_o(y)-R_o(y))-\frac{\pi}{4y^{\frac12}}$ is skew-self reciprocal in the transform of order (0, 0)

(i) Let

$$f(y)=\left\{\frac{\pi}{2}\,y^{\frac12}(I_o(y)-L_o(y))-\frac{1}{2y^{\frac12}}\right\}\,;\ \mu=\nu=0$$

Then by applying Hankel's Theorem and on simplification, we get

$$
\begin{aligned}
x^{\frac12}\cdot\int_0^\infty J_o(t)\,t^{-1}\,dt\int_0^\infty J_o(xy/t)\{\tfrac12\pi\, y(I_o(y)-L_o(y))-\tfrac12\}\,dy
&=\frac{1}{x^{\frac12}}\int_0^\infty\frac{t^2J_o(t)\,dt}{x^2+t^2}-\frac{1}{2x^{\frac12}}\\
&=\frac{1}{x^{\frac12}}\int_0^\infty J_o(t)\left\{1-\frac{x^2}{x^2+t^2}\right\}dt-\frac{1}{2x^{\frac12}}\\
&=\frac{1}{2x^{\frac12}}-\tfrac12\pi x^{\frac12}(I_o(x)-L_o(x))\,; \qquad (11)
\end{aligned}
$$

showing that $\frac12\pi y^{\frac12}(I_o(y)-L_o(y))-\frac12 y^{\frac12}$ is skew-self-reciprocal under this transform of order (0, 0). We might also note here that $x^{-\frac12}$ is also self-reciprocal under this transform.

The following conditions [*Hobson* (*1907*), *p. 540*] *are sufficient for the equality of the repeated integrals*:

$$\int_a^\infty dx \int_c^\infty f(x, y)\, dy, \int_c^\infty dy \int_a^\infty f(x, y)\, dx;$$

(1) *that the repeated integrals between finite limits exist and are equal*,

(2) *that* $\int_a^\infty f(x, y)\, dx$ *be uniformly convergent in an arbitrary interval*,

(3) *that* $\int_c^\infty f(x, y)\, dy$ *satisfies the same condition, and*

(4) $\int_a^\infty dy \int_c^Y f(x, y)\, dy$ *converges uniformly in an unlimited interval.*

The conditions are satisfied in the above cases, showing that change in the order of integration is permissible.

3. We shall now obtain certain formal solutions of the problem

Let $\tau(s)$ be the Mellin Transform (Titchmarsh, *loc. cit.*, pp 7, 46, 53) of $f(x)$. Then we have

$$\tau(s) = \int_0^\infty x^{s-1} f(x)\, dx, \tag{A}$$

where

$$f(x) = \int_0^\infty \varpi_{\mu,\nu}(x, y)\, f(y)\, dy,$$

and

$$\varpi_{\mu,\nu}(xy) = (xy)^{\frac{1}{2}} \int_0^\infty J_\nu(t)\, J_\mu(xy/t)\, t^{-1} dt$$

Hence

$$\begin{aligned}
\tau(s) &= \int_0^\infty x^{s-1} dx \int_0^\infty \varpi_{\mu,\nu}(xy)\, f(y) dy \\
&= \int_0^\infty f(y) dy \int_0^\infty x^{s-1} \varpi_{\mu,\nu}(xy)\, dx \\
&= \int_0^\infty y^{\frac{1}{2}} f(y)\, dy \left\{ \int_0^\infty \int_0^\infty x^{s-\frac{1}{2}}\, J_\nu(t)\, J_\mu(xy/t)\, t^{-1} dt\, dx \right\} \\
&= \int_0^\infty y^{\frac{1}{2}} f(y)\, dy \left\{ \int_0^\infty J_\nu(t)\, t^{-1} dt \int_0^\infty x^{s-\frac{1}{2}}\, J_\mu(xy/t)\, dx \right\} \\
&= 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\mu + \frac{1}{2}s + \frac{1}{4})}{\Gamma(\frac{1}{2}\mu - \frac{1}{2}s + \frac{3}{4})} \int_0^\infty y^{-s} f(y) dy \int_0^\infty J_\nu(t)\, t^{s-\frac{1}{2}} dt \\
&= 2^{2s-1} \frac{\Gamma(\frac{1}{2}\mu + \frac{1}{2}s + \frac{1}{4})\, \Gamma(\frac{1}{2}\nu + \frac{1}{2}s + \frac{1}{4})}{\Gamma(\frac{1}{2}\mu - \frac{1}{2}s + \frac{3}{4})\, \Gamma(\frac{1}{2}\nu - \frac{1}{2}s + \frac{3}{4})}\, \tau(1-s);
\end{aligned}$$

then if we assume

$$\tau(s) = 2^{s}\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\,\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\,\psi(s);$$

we get

$$\psi(s) = \psi(1-s). \tag{12}$$

We thus obtain a general solution of (A); viz

$$f(x) = \frac{1}{2\pi i}\int_{c+i\infty}^{c+i\infty} 2^{s}\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\,\psi(s)x^{-s}\,ds \tag{B}$$

where $\psi(s)$ satisfies (2).

Following Hardy and Titchmarsh (*loc cit.*, pp. 246, 252, 53) we shall say that $f(x)$ belongs to $A(\alpha, a)$ where $0 < \alpha \leq \pi$, $a < \frac{1}{2}$, if it is analytic and regular in the region defined by $r < 0$, $|\theta < \alpha$ and if it is $O(|x|^{-a-\epsilon})$ for small x and $O(|x|^{a-1+\epsilon})$ for large x, for every positive ϵ and uniformly in the angle $\theta| \leq \alpha-\eta < \alpha$.

The necessary and sufficient condition that $f(x)$ of $A(\alpha, a)$ should be its own $R_{\mu,\nu}$ transform is that it should be of the form (B). Bhatnagar (*a*) has proved that the necessary and sufficient condition that $f(x)$ of $A(\alpha, a)$ should be its own $R_{\mu,\nu}$ transform is that it should be of the form (B) where $\psi(s)$ is regular and satisfies the relation $\psi(s) = \psi(1-s)$ in the strip $a < \sigma < 1-a$ and is of the order $O(e^{(\pi/2-\alpha+\eta)|t|})$, $\theta| \leq \alpha-\eta < \alpha$.

Another formal solution of the problem is obtained by assuming

$$\begin{aligned}
J(s) &= \int_0^\infty (xs)^{\frac{1}{2}} K_0(xs)\, f(x)\, dx; \qquad \mu = \nu = 0\\
&= \int_0^\infty (xs)^{\frac{1}{2}} K_0(xs)\, dx \int_0^\infty \varpi_{0,0}(xy)\, f(y)\, dy\\
&= \int_0^\infty f(y)\, dy \left\{ \int_0^\infty (xs)^{\frac{1}{2}} K_0(xs)\, dx \int_0^\infty (xy)^{\frac{1}{2}} J_0(t)\, J_0(xy/t)\, \frac{dt}{t}\right.\\
&= \int_0^\infty (ys)^{\frac{1}{2}} f(y)\, dy \int_0^\infty J_0(t)\, t^{-1}\, dt/(y^2/t^2+s^2)
\end{aligned}$$

Hence

$$\begin{aligned}
J(s) &= \int_0^\infty y^{\frac{1}{2}} s^{-3/2} K_0(y/s)\, f(y)\, dy\\
&= s^{-1} J(1/s)
\end{aligned} \tag{13}$$

4. We shall say $f(x)$ belongs to $R_{\mu,\nu}$ if $f(x)$ satisfies the equation

$$f(x) = \int_0^\infty \varpi_{\mu,\nu}(xy)\, f(y)\, dy$$

If $f(x)$ belongs to $R_{\mu,\nu}$ and

$$K(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{2s-1}\{\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\,\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\}^2\,\chi(s)x^{-s}\,ds$$

where $\chi(s) = \chi(1-s)$;

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then

$$g(s) = \int_0^\infty f(y)\, K(xy)\, dy \text{ belongs to } R_{\mu,\nu}$$

A general formula for $f(x)$ of $R_{\mu,\nu}$ is

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^s \Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}s + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s + \tfrac{1}{4})\, \psi(s) x^{-s} ds,$$

where $\psi(s) = \psi(1-s)$.

Then (Titchmarsh, *loc cit.*, pp. 246, 252, 53.)

$$g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^s \Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}s + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s + \tfrac{1}{4})\, \psi_1(s) x^{-s} ds$$

where

$$\psi_1(s) = \Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}s + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}\mu + \tfrac{3}{4} - \tfrac{1}{2}s)\, \Gamma(\tfrac{1}{2}\nu + \tfrac{3}{4} - \tfrac{1}{2}s)\, \psi(1-s)\, \chi(s)$$

and

$$\psi_1(s) = \psi_1(1-s)$$

Let

$$K(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}s + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}\mu + \tfrac{3}{4} - \tfrac{1}{2}s)\, \Gamma(\tfrac{1}{2}\nu + \tfrac{3}{4} - \tfrac{1}{2}s)\, \chi(s)\, x^{-s} ds$$

where $\chi(s) = \chi(1-s)$.

Then

$$g(x) = \frac{1}{x} \int_0^\infty f(y) K\left(\frac{y}{x}\right) dy \text{ belongs to } R_{\mu,\nu};$$

for

$$g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^s \Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}s + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s + \tfrac{1}{4})\, \psi_2(s) x^{-s}\, ds$$

where

$$\psi_2(s) = \Gamma\tfrac{1}{2}\mu + \tfrac{1}{2}s + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}\mu + \tfrac{3}{4} - \tfrac{1}{2}s)\, \Gamma(\tfrac{1}{2}\nu + \tfrac{3}{4} - \tfrac{1}{2}s)\, \chi(1-s)\, \psi(s)$$

and

$$\psi_2(s) = \psi_2(1-s)$$

Also we have

$$K\left(\frac{1}{x}\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}s + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}s + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}\mu + \tfrac{3}{4} - \tfrac{1}{2}s)\, \Gamma(\tfrac{1}{2}\nu + \tfrac{3}{4} - \tfrac{1}{2}s)\, \chi(s)\, x^s\, ds$$

This is equivalent to

$$K(1/x) = xK(x) \tag{14}$$

5. Bhatnagar (a) has proved that the sufficient condition that a function shall be $R_{\mu,\nu}$ is that

$$\int_0^\infty (xt)^{\frac{1}{2}} J_\nu(xt)\, f(t) dt = x^{-\frac{1}{2}} \int_0^\infty y^{\frac{1}{2}} J_\mu(y/x)\, f(y)\, dy,$$

$$(\mu,\ \nu > -\tfrac{1}{2})$$

If $\mu = \nu$, $\varphi(x) = x^{-1}\, \varphi(1/x)$, where

$$\varphi(x) = \int_0^\infty (xt)^{\frac{1}{2}} J_\nu(xt)\, f(t)\, dt$$

and

$$f(x) = \int_0^\infty (xt)^{\frac{1}{2}} J_\nu(xt)\, \varphi(t)\, dt$$

$f(x)$ is assumed to be continuous and absolutely integrable in $(0, \infty)$. We are thus able to find $f(x)$ which shall be $R_{\nu,\nu}$.

6. Let

$$f(x) = x^{\frac{1}{2}}\{K_0(x) - \tfrac{1}{9}K_0(\tfrac{1}{3}x) - \tfrac{1}{25}K_0(\tfrac{1}{5}x) + \ldots\} ;$$

$$g(x) = x^{\frac{1}{2}}\{K_0(x) - K_0(3x) - K_0(5x) + \ldots\} ;$$

$$\mu = \nu = 0.$$

Then $f(x)$ and $g(x)$ are reciprocal under this transform.

We have

$$\begin{aligned}
&\int_0^\infty \varpi_{0,0}(xy)\, f(y)\, dy \\
&= x^{\frac{1}{2}} \int_0^\infty J_0(t)\, t^{-1} dt \int_0^\infty J_0(xy/t) y \{K_0(y) - \tfrac{1}{9}K_0(3y) - \tfrac{1}{25}K_0(\tfrac{1}{5}y) \ldots\} dy \\
&= x^{\frac{1}{2}} \int_0^\infty J_0(t)\, t \left\{ \frac{1}{x^2+t^2} - \frac{1}{9x^2+t^2} - \frac{1}{25x^2+t^2} + \ldots \right\} dt \\
&= x^{\frac{1}{2}}\{K_0(x) - K_0(3x) - K_0(5x) + \ldots\} \\
&= g(x).
\end{aligned}$$

Again

$$\begin{aligned}
&\int_0^\infty \varpi_{0,0}(xy)\, g(y)\, dy \\
&= x^{\frac{1}{2}} \int_0^\infty J_0(t)\, t^{-1} dt \int_0^\infty J_0(xy/t) \{K_0(y) - K_0(3y) - K_0(5y) + \ldots\} dy \\
&= x^{\frac{1}{2}} \int_0^\infty J_0(t)\, t \left\{ \frac{1}{x^2+t^2} - \frac{1}{x^2+9t^2} - \frac{1}{x^2+25t^2} + \ldots \right\} dt \\
&= x^{\frac{1}{2}}\{K_0(x) - \tfrac{1}{9}K_0(\tfrac{1}{3}x) - \tfrac{1}{25}K_0(\tfrac{1}{5}x) + \ldots\} \\
&= f(x)
\end{aligned}$$

Thus if we have,

if

$$g(x) = \int_0^\infty \varpi_{0,0}(xy)\, f(y)\, dy$$

then

$$f(x) = \int_0^\infty \varpi_{0,0}(xy)\, g(y)\, dy$$

As regards change in the orders of integrations and term-by-term integrations of the infinite series, the senior author has dwelt upon it at length in a previous paper (Mitra, 1949). The procedure is exactly the same.

The generalised transform is

$$\varpi_{\nu_1,\nu_2,\ldots,\nu_n}(xy) = (xy)^{\frac{1}{2}} \int_0^\infty \ldots \int_0^\infty J_{\nu_1}(t_1) J_{\nu_2}(t_2) \ldots J_{\nu_{n-1}}(t_{n-1}) J_{\nu_n}\!\left(\frac{xy}{t_1 \ldots t_{n-1}}\right) dt_1 \ldots dt_{n-1}/t_1 t_2 \ldots t_{n-1}$$

Bhatnagar (*b*) has proved in *details* that the kernel $\varpi_{\mu,\nu}(xy)$ plays the role of a transform Not only this, but if we define by

$$\varpi_{\mu,\nu,\lambda}(xy) = (xy)^{\frac{1}{2}}\int_0^\infty\int_0^\infty J_\lambda(t)\,J_\nu(y)\,J_\mu\left(\frac{xz}{yt}\right)\frac{dy\,dt}{yt},$$

$$R(\mu+\tfrac{1}{2})\geq 0,\ R(\nu+\tfrac{1}{2})\geq 0,\ R(\lambda+\tfrac{1}{2})\geq 0$$

then $\varpi_{\mu,\nu,\lambda}(xy)$ also plays the role of a transform

This has been investigated by Bhatnagar (*b*) and a generalisation of the transform can be effected. The generalised transform is being studied by some of the collaborators of the senior author. The theoretical transform $R_{\mu,\nu}$ has been dealt with at length by Bhatnagar.

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NOTE ON SYSTEMS OF EQUI-POTENTIAL *N*-SURFACES (ESPECIALLY, *N*-ELLIPSOIDS) IN AN (*N*+1)-SPACE

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Abstract

This short paper begins primarily with the Plückerian concept of a focus, as adapted to an n-surface immersed in an $(n+1)$-space; this is followed by a brief reference to the $(n+1)$ component "*focal* $(n-1)$-*surfaces*" of an n-ellipsoid, and a broad survey of a system of confocal n-ellipsoids. Next, the condition for an assigned set of n-surfaces to represent an isometric (or equi-potential) system has been investigated and then applied in the verification of the proposition that a family of confocal n-ellipsoids is isometric. In short, the subject-matter of this paper can be described as a generalisation to multi-dimensional space of a few cardinal propositions, (of a 3-space), bearing on *isometric* systems of surfaces (*e.g.* confocal quadrics). The generalisations effected are believed to be new.

1. Given an n-surface Γ, immersed in an $(n+1)$-space, we may after Plücker define a focus S of Γ as a point, the infinitesimal n-sphere at which has double contact with Γ. This definition automatically leads to *two* independent relations among the $(n+1)$ coordinates of S, shewing that Γ possesses, in general, an aggregate of ∞^{n-1} foci, whose locus is a certain $(n-1)$-surface, (say, Ω). This locus, which is generally composed of a finite number of detached portions, will be designated in this paper as the "*focal* $(n-1)$-*surface*", attaching to the given n-surface Γ.

We shall now confine our attention to the special case when Γ happens to be a central n-quadric, whose Cartesian equation, referred to its $(n+1)$ principal axes, is

$$\sum_{r=0}^{r=n} x_r^2/a_r = 1. \tag{1}$$

Generalising the traditional method of investigating the foci of an ordinary central quadric of a 3-space, we find without much ado that the *complete* "focal $(n-1)$-surface" of Γ consists of $(n+1)$ component $(n-1)$-surfaces, lying respectively in the $(n+1)$ principal n-flats (of Γ). The pairs of (Cartesian) equations, representing the $(n+1)$ component loci, are easily seen to be

$$\text{(I)}\quad x_0 = 0,\quad \text{coupled with } \frac{x_1^2}{a_1-a_0}+\frac{x_2^2}{a_2-a_0}+\frac{x_3^2}{a_3-a_0}+\cdots+\frac{x_n^2}{a_n-a_0}=1\ ;$$

$$\text{(II)}\quad x_1 = 0,\quad \text{coupled with } \frac{x_0^2}{a_0-a_1}+\frac{x_2^2}{a_2-a_1}+\frac{x_3^2}{a_3-a_1}+\cdots+\frac{x_n^2}{a_n-a_1}=1\ ;$$

$$\text{(III)}\quad x_2 = 0,\quad \text{coupled with } \frac{x_0^2}{a_0-a_2}+\frac{x_1^2}{a_1-a_2}+\frac{x_3^2}{a_3-a_2}+\cdots+\frac{x_n^2}{a_n-a_2}=1\ ;$$

. . . .

and so on.

The logical conclusion is that the most general type of (central) n-quadric, *confocal* with the original central n-quadric Γ, as defined by (1), is

$$\sum_{r=0}^{r=n} \frac{x_r^2}{a_r+\lambda} = 1, \tag{3}$$

where λ is a *variable* parameter.

It is now a tame affair to verify that the majority of commonplace lemmas and theorems on a system of confocal quadrics of an ordinary 3-space must, when properly generalised, hold good for a system of confocal n-quadrics, *viz.* (3), lying in an $(n+1)$ space. Thus, for instance, the total number of confocal n-quadrics of the type (3), that pass through an *assigned* point is, in general, $n+1$, and of these only one will be an n-ellipsoid.[†]

In the present paper we are particularly interested in a family of *confocal n-ellipsoids*, which will be shewn to constitute a family of *isometric* or *equipotential* n-surfaces. To that end we have to investigate (in the next article) a *general* method of testing whether a *given* family of n-surfaces is isometric or not.

2. Suppose that a system of n-surfaces, situated in an $(n+1)$ space, is given in the symbolic form:

$$\lambda = \text{const.}, \tag{1}$$

where λ is a known function* of the $(n+1)$ current Cartesian coordinates $(x_0, x_1, x_2, \ldots, x_n)$ and the right-hand side constant is a *variable* parameter.

Plain reasoning points to the conclusion that, in order that (1) may define a set of *isometric* n-surfaces, the necessary and sufficient condition is that some function V of λ *viz*:

$$V = \varphi(\lambda), \tag{2}$$

should be a "hyper-spherical harmonic," *i.e.*, should conform to the generalised Laplacian equation, *viz*:

$$\nabla^2 V = 0, \tag{3}$$

where

$$\nabla^2 \equiv \sum_{r=0}^{r=n} \left(\frac{\partial}{\partial x_r}\right)^2.$$

In view of the differentiated results of the type:

$$\frac{\partial^2 V}{\partial x_r^2} = \varphi'(\lambda)\,\frac{\partial^2 \lambda}{\partial x_r^2} + \varphi''(\lambda).\left(\frac{\partial \lambda}{\partial x_r}\right)^2, \quad (r = 0, 1, 2, 3, \ldots, n),$$

(3) is plainly equivalent to

$$\varphi'(\lambda)\nabla^2\lambda + \varphi''(\lambda).\sum_{r=0}^{r=n}\left(\frac{\partial \lambda}{\partial x_r}\right)^2 = 0. \tag{4}$$

[†] An n-quadric like $\sum_{r=0}^{r=n} x_r^2/a_r = 1$ is called an *n-ellipsoid*, when $a_r > 0$ for *every* r.

* It is presumed that λ is a known continuous function of the $(n+1)$ real variables $(x_0, x_1, x_2, \ldots x_n)$ and admits of continuous partial differential coefficients of at least the second order with respect to each of them.

We thus arrive at the following proposition :—

The condition—at once necessary and sufficient—for the equation (1) to represent a family of equi-potential n-surfaces is that the quantity

$$\nabla^2\lambda \Big/ \Big[\sum_{r=0}^{r=n} \Big(\frac{\partial\lambda}{\partial x_r}\Big)^2\Big] \tag{5}$$

be expressible solely as a function of λ *(say,* $\chi(\lambda)$*). Subject to this condition, the actual expression* $\varphi(\lambda)$ *for the potential function V, attaching to an individual n-surface* (λ) *of the system, is determinable by the differential equation :*

$$\frac{\varphi''(\lambda)}{\varphi'(\lambda)} = -\chi(\lambda). \tag{6}$$

Admittedly the method outlined as above is nothing more than a simple adaptation of the time-honoured method, often adopted in the treatment of the corresponding problem of three dimensions [*Cf.* Forsyth (1912), Ramsey (1940)].

3. As an application of the proposition established in the fore-going article, let us start with a system of confocal n-ellipsoids in the form :

$$\lambda = \text{const.}, \tag{1}$$

where λ is a function of the $(n+1)$ rectangular Cartesian coordinates $(x_0, x_1, x_2, \ldots, x_n)$, satisfying the equation :

$$\sum_{r=0}^{r=n} x_r^2/(a_r+\lambda) = 1. \tag{2}$$

If for once we pick out at random one of the x's (say, x_r) and differentiate λ with respect to x_r, and further introduce the abridged notations :

$$P \equiv \sum_{r=0}^{r=n} x_r^2/(a_r+\lambda)^2 \quad \text{and} \quad Q \equiv \sum_{r=0}^{r=n} x_r^2/(a_r+\lambda)^3, \tag{3}$$

we find without much labour

$$\frac{\partial\lambda}{\partial x_m} = \frac{2}{P}\cdot\frac{x_m}{a_m+\lambda}, \tag{4}$$

and

$$\frac{\partial P}{\partial x_m} = \frac{2x_m}{(a_m+\lambda)^2} - 2Q.\frac{\partial\lambda}{\partial x_m}. \tag{5}$$

Consequently,

$$\begin{aligned}\frac{\partial^2\lambda}{\partial x_m^2} &= \frac{2}{P}.\frac{1}{a_m+\lambda} - \frac{2}{P^2}.\frac{x_m}{a_m+\lambda}.\frac{\partial P}{\partial x_m} - \frac{2}{P}.\frac{x_m}{(a_m+\lambda)^2}.\frac{\partial\lambda}{\partial x_m}\\ &= \frac{2}{P}.\frac{1}{a_m+\lambda} - \frac{4}{P^2}.\frac{x_m^2}{(a_m+\lambda)^3} + \frac{\partial\lambda}{\partial x_m}.\Big\{\frac{4Q}{P^2}.\frac{x_m}{a_m+\lambda} - \frac{2}{P}.\frac{x_m}{(a_m+\lambda)^2}\Big\}, \quad \text{by} \quad (5)\\ &= \frac{2}{P}.\frac{1}{a_m+\lambda} - \frac{8}{P^2}.\frac{x_m^2}{(a_m+\lambda)^3} + \frac{8Q}{P^3}.\frac{x_m^2}{(a_m+\lambda)^2}, \quad \text{by} \quad (4)\end{aligned}$$

If we now put

$$R \equiv \sum_{r=0}^{r=n} \frac{1}{a_r+\lambda}, \tag{6}$$

we readily obtain

$$\nabla^2\lambda = \sum_{m=0}^{m=n} \frac{\partial^2\lambda}{\partial {x_m}^2} = \frac{2R}{P} - \frac{8}{P^3}\cdot\sum_{m=0}^{m=n} \frac{{x_m}^2}{(a_m+\lambda)^3} + \frac{8Q}{P^3}\cdot\sum_{m=0}^{m=n} \frac{{x_m}^2}{(a_m+\lambda)^2}$$

$$= \frac{2R}{P} - \frac{8}{P^2}\cdot Q + \frac{8Q}{P^2}\cdot P, \quad \text{by (3) and (6)}$$

$$= \frac{2R}{P}.$$

Moreover we find from (4)

$$\sum_{m=0}^{m=n} \left(\frac{\partial\lambda}{\partial x_m}\right)^2 = \frac{4}{P^2}\cdot\sum_{m=0}^{m=n} \frac{{x_m}^2}{(a_m+\lambda)^2} = \frac{4}{P}, \quad \text{by (3).}$$

Evidently, then, we have

$$\nabla^2\lambda \Big/ \sum_{r=0}^{r=n} \left(\frac{\partial\lambda}{\partial x_r}\right)^2 = \frac{1}{2}\cdot R = \frac{1}{2}\cdot\sum_{r=0}^{r=n} \frac{1}{a_r+\lambda} \equiv \text{a function of } \lambda$$

$$= \chi(\lambda), \quad \text{(say).} \tag{7}$$

Hence, on the strength of Art. 2, we conclude that the confocal n-ellipsoids, *viz*: (2) make up an *equi-potential* system. To calculate the potential $V = \varphi(\lambda)$, associated with the n-ellipsoid 'λ', we appeal to the concluding paragraph of Art. 2. This gives

$$\frac{\varphi''(\lambda)}{\varphi'(\lambda)} = -\chi(\lambda) = -\frac{1}{2}\cdot\sum_{r=0}^{r=n} \frac{1}{(a_r+\lambda)}.$$

Manifestly the complete primitive is

$$V = \varphi(\lambda) = C\int_k^\lambda \frac{dt}{\sqrt{\{(a_0+t)(a_1+t)\cdots(a_n+t)\}}}, \tag{8}$$

where C and k are the two constants of integration.

Putting this and that together, we reach the final conclusion that *the confocal n-ellipsoids viz*: (2) *constitute an equipotential system, the actual value of the potential V, associated with any such surface, being determined by* (8).

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INTER-RELATIONS OF PATHS AND AFFINE CONNECTIONS IN A NON-RIEMANNIAN SPACE

BY

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1. **Introduction.** Eisenhart (1927) has found out the equations of the path corresponding to a projectively flat space. But he has not taken up the general question of determining the path when the affine connections of a non-Riemannian space, are given. In this paper the reverse process of determining the affine connections, when the paths are given, is taken into consideration. The object set forth is to investigate if there always exists an affine connection corresponding to any arbitrary form of path taken in a non-Riemannian space.

2. **The Affine Connections**—Banesh Hoffmann (1932) has put the equations to a geodesic namely

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\cdot\frac{dx^\gamma}{ds}=0,\quad (\alpha=1,2,3,4), \tag{1}$$

in the form

$$\frac{d^2x^\alpha}{dt^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\cdot\frac{dx^\gamma}{dt}-\Gamma^4_{\beta\gamma}\frac{dx^\beta}{dt}\cdot\frac{dx^\gamma}{dt}\cdot\frac{dx^\alpha}{dt}=0,\quad (\alpha=1,2,3), \tag{1-a}$$

where t has been written for x^4. He has considered a hypothetical experiment in which particles and light pulse emanating from a bomb explosion, are observed and $\frac{dx^\alpha}{dt}$, $\frac{d^2x^\alpha}{dt^2}$, $\frac{d^3x^\alpha}{dt^3}$, etc. for a particle or a pulse are computed at the starting point from the observed values of x, y, z, t along its trajectory. Then $\frac{dx^\alpha}{dt}$, $\frac{d^2x^\alpha}{dt^2}$, etc. at other points are expressed as Traylor's series about the point of explosion supposing the circumstances for such an expansion to hold good. Thus he has shown that by observing a sufficient number of trajectories and substituting for $\frac{dx^\alpha}{dt}$, $\frac{d^2x^\alpha}{dt^2}$, etc in $(1-a)$, $\Gamma^\alpha_{\beta\gamma}$ can be expressed as functions of coordinates. But this method fails in the more general case of paths and affine connections belonging to a non-Riemannian space for the simple reason that $\frac{dx^\alpha}{dt}$, $\frac{d^2x^\alpha}{dt^2}$, etc cannot be expressed as functions of coordinates. However if, as Eisenhart (1927) has done, we express the equations to the path in a form where t is not the independent variable and x, y, z, t are functions of s where s is the affine parameter then it is not necessary to throw the equations of the path from the form (1) to the form (1-a) and $\Gamma^\alpha_{\beta\gamma}$ can be expressed as functions of the coordinates by substituting for

$\frac{dx^a}{ds}$, $\frac{d^2x^a}{ds^2}$, in (1). (No confusion needs rise here even if the form of the equations to the path in a non-Riemannian manifold is the same as (1) for the symbols change their meanings with manifold). It is a different matter if this procedure fails and we are not able to express $\Gamma^a_{\beta\gamma}$ in terms of co-ordinates for the simple reason that there is no affine connection corresponding to the path taken. It will be illustrated in the following by taking different cases in a non-Riemannian 3-dimensional space namely x, y, t.

Case (*i*), If the equations to the path are of the form

$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{t-t_0}{n} = \varphi(s) \tag{2}$$

where x_0, y_0, t_0, l, m, n are arbitrary constants and s is the affine parameter, the different differential coefficients are given by

$$\frac{dx}{ds} = l\frac{d\varphi}{ds},\ \frac{dy}{ds} = m\frac{d\varphi}{ds},\ \frac{dt}{ds} = n\frac{d\varphi}{ds}, \tag{3}$$

$$\frac{d^2x}{ds^2} = l\,\frac{d^2\varphi}{ds^2},\quad \frac{d^2y}{ds^2} = m\,\frac{d^2\varphi}{ds^2},\quad \frac{d^2t}{ds^2} = n\,\frac{d^2\varphi}{ds^2}. \tag{4}$$

Substituting these in one of the equations of (1), the following is obtained;

$$l\,\frac{d^2\varphi}{ds^2} + \Gamma^1_{11}l^2\left(\frac{d\varphi}{ds}\right)^2 + \Gamma^1_{22}m^2\left(\frac{d\varphi}{ds}\right)^2 + \Gamma^1_{44}n^2\left(\frac{d\varphi}{ds}\right)^2$$

$$+2\Gamma^1_{12}lm\left(\frac{d\varphi}{ds}\right)^2 + 2\Gamma^1_{14}ln\left(\frac{d\varphi}{ds}\right)^2 + 2\Gamma^1_{24}mn\left(\frac{d\varphi}{ds}\right)^2 = 0. \tag{5}$$

Since l, m, n are arbitrary, it can be easily shown that

$$\left.\begin{array}{l}\Gamma^1_{11} = 0,\ \ \Gamma^1_{22} = 0,\ \ \Gamma^1_{44} = 0,\\ \Gamma^1_{12} = 0,\ \ \Gamma^1_{14} = 0,\ \ \Gamma^1_{24} = 0.\end{array}\right\} \tag{6}$$

Similarly from the other equations of the path it can be shown that

$$\Gamma^2_{\beta\gamma} = 0, \tag{7}$$

$$\Gamma^4_{\beta\gamma} = 0. \tag{8}$$

Thus if the form of the path is given by (2), the corresponding space must have $\Gamma^a_{\beta\gamma}$'s $= 0$

Case (*ii*). Let the equations of the path be of the form

$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{t-t_0}{n} = r \tag{9}$$

where x_0, y_0, t_0, r are arbitrary constants and l, m, n satisfy the relation,

$$Al+Bm+Cn = 0 \tag{10}$$

where A, B, C are arbitrary constants. The different differential coefficients are given by

$$\frac{dx}{ds} = r\frac{dl}{ds},\ \frac{dy}{ds} = r\,\frac{dm}{ds},\ \frac{dt}{ds} = r\frac{dn}{ds} \tag{11}$$

From (10) we get

$$A\frac{dl}{ds}+B\frac{dm}{ds}+C\frac{dn}{ds}=0. \tag{12}$$

$$A\frac{d^2l}{ds^2}+B\frac{d^2m}{ds^2}+C\frac{d^2n}{ds^2}=0. \tag{13}$$

Substituting these in one of the equations of (1), the following equation is obtained;

$$r\frac{d^2l}{ds^2}+\Gamma^1_{11}r^2\left(\frac{dl}{ds}\right)^2+\Gamma^1_{22}r^2\left(\frac{dm}{ds}\right)^2+\Gamma^1_{44}r^2\left(\frac{dn}{ds}\right)^2$$
$$+2\Gamma^1_{12}r^2\frac{dl}{ds}\cdot\frac{dm}{ds}+2\Gamma^1_{14}r^2\frac{dl}{ds}\cdot\frac{dn}{ds}+2\Gamma^1_{24}r^2\frac{dm}{ds}\cdot\frac{dn}{ds}=0. \tag{14}$$

Since r is arbitrary, the following must hold good;

$$\Gamma^1_{11}+\Gamma^1_{22}\left(\frac{dm}{ds}\right)^2\Big/\left(\frac{dl}{ds}\right)^2+\Gamma^1_{44}\left(\frac{dn}{ds}\right)^2\Big/\left(\frac{dl}{ds}\right)^2+2\Gamma^1_{12}\frac{dm}{ds}\Big/\frac{dl}{ds}$$
$$+2\Gamma^1_{14}\frac{dn}{ds}\Big/\frac{dl}{ds}+2\Gamma^1_{24}\left(\frac{dm}{ds}\Big/\frac{dl}{ds}\right)\left(\frac{dn}{ds}\Big/\frac{dl}{ds}\right)=0. \tag{15}$$

From (12), it is obtained that

$$\frac{dn}{ds}\Big/\frac{dl}{ds}=-\frac{1}{C}\left(A+B\frac{dm}{ds}\Big/\frac{dl}{ds}\right). \tag{16}$$

Substituting this in (15), it is obtained that

$$\Gamma^1_{11}+\Gamma^1_{22}\left(\frac{dm}{ds}\right)^3\Big/\left(\frac{dl}{ds}\right)^3+\Gamma^1_{44}\frac{1}{C^2}\left(A+B\frac{dm}{ds}\Big/\frac{dl}{ds}\right)^2+2\Gamma^1_{12}\frac{dm}{ds}\Big/\frac{dl}{ds}$$
$$-\frac{2}{C}\Gamma^1_{14}\left(A+B\frac{dm}{ds}\Big/\frac{dl}{ds}\right)-\frac{2}{C}\Gamma^1_{21}\left(\frac{dm}{ds}\Big/\frac{dl}{ds}\right)\left(A+B\frac{dm}{ds}\Big/\frac{dl}{ds}\right)=0. \tag{17}$$

Since A, B, C are arbitrary, it can be easily shown that

$$\Gamma^1_{44}=0,\quad \Gamma^1_{22}=0; \tag{18}$$

$$\Gamma^1_{11}\left(\frac{dl}{ds}\right)^2=-2\Gamma^1_{12}\frac{dl}{ds}\cdot\frac{dm}{ds}=-2\Gamma^1_{14}\frac{dl}{ds}\cdot\frac{dn}{ds}=2\Gamma^1_{24}\frac{dm}{ds}\frac{dn}{ds} \tag{19}$$

Similar is the case with $\Gamma^2_{\beta\gamma}$, $\Gamma^4_{\beta\gamma}$.

So it is seen that $\Gamma^a_{\beta\gamma}$ cannot be expressed in terms of the coordinates. Hence there exists no space with an affine connection such that the equations of the path can be as taken in this case.

Case (*iii*). If the equations of the path are of the form

$$\frac{x-x_0}{l+\varepsilon_1}=\frac{y-y_0}{m+\varepsilon_2}=\frac{t-t_0}{n+\varepsilon_4}=\varphi(s) \tag{20}$$

where x_0, y_0, t_0, l, m, n are arbitrary constants and φ is a function of the affine parameter s, it is to be investigated whether a metric can exist under any condition after determining the expressions for $\Gamma^{\alpha}_{\beta\gamma}$'s. The different differential coefficients are given by

$$\left.\begin{aligned} \frac{dx}{ds} &= (l+\varepsilon_1)\frac{d\varphi}{ds} + \frac{d\varepsilon_1}{ds}\,\varphi, \\ \frac{dy}{ds} &= (m+\varepsilon_2)\frac{d\varphi}{ds} + \frac{d\varepsilon_2}{ds}\,\varphi, \\ \frac{dt}{ds} &= (n+\varepsilon_4)\frac{d\varphi}{ds} + \frac{d\varepsilon_4}{ds}\,\varphi \end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned} \frac{d^2x}{ds^2} &= (l+\varepsilon_1)\frac{d^2\varphi}{ds^2} + 2\frac{d\varepsilon_1}{ds}\cdot\frac{d\varphi}{ds} + \frac{d^2\varepsilon_1}{ds^2}\,\varphi, \\ \frac{d^2y}{ds^2} &= (m+\varepsilon_2)\frac{d^2\varphi}{ds^2} + 2\frac{d\varepsilon_2}{ds}\cdot\frac{d\varphi}{ds} + \frac{d^2\varepsilon_2}{ds^2}\,\varphi, \\ \frac{d^2t}{ds^2} &= (n+\varepsilon_4)\frac{d^2\varphi}{ds^2} + 2\frac{d\varepsilon_4}{ds}\cdot\frac{d\varphi}{ds} + \frac{d^2\varepsilon_4}{ds^2}\,\varphi. \end{aligned}\right\} . \tag{22}$$

Substituting these in an equation of (1), the following equation is obtained:

$$\begin{aligned} &(l+\varepsilon_1)\frac{d^2\varphi}{ds^2} + 2\frac{d\varepsilon_1}{ds}\cdot\frac{d\varphi}{ds} + \frac{d^2\varepsilon_1}{ds^2}\cdot\varphi + \Gamma^1_{11}\left\{(l+\varepsilon_1)\frac{d\varphi}{ds} + \frac{d\varepsilon_1}{ds}\cdot\varphi\right\}^2 \\ &\quad + \Gamma^1_{22}\left\{(m+\varepsilon_2)\frac{d\varphi}{ds} + \frac{d\varepsilon_2}{ds}\cdot\varphi\right\}^2 + \Gamma^1_{44}\left\{(n+\varepsilon_4)\frac{d\varphi}{ds} + \frac{d\varepsilon_4}{ds}\cdot\varphi\right\}^2 \\ &\quad + 2\Gamma^1_{12}\left\{(l+\varepsilon_1)\frac{d\varphi}{ds} + \frac{d\varepsilon_1}{ds}\cdot\varphi\right\}\left\{(m+\varepsilon_2)\frac{d\varphi}{ds} + \frac{d\varepsilon_2}{ds}\cdot\varphi\right\} \\ &\quad + 2\Gamma^1_{14}\left\{(l+\varepsilon_1)\frac{d\varphi}{ds} + \frac{d\varepsilon_1}{ds}\,\varphi\right\}\left\{(n+\varepsilon_4)\frac{d\varphi}{ds} + \frac{d\varepsilon_4}{ds}\cdot\varphi\right\} \\ &\quad + 2\Gamma^1_{24}\left\{(m+\varepsilon_2)\frac{d\varphi}{ds} + \frac{d\varepsilon_2}{ds}\cdot\varphi\right\}\left\{(n+\varepsilon_4)\frac{d\varphi}{ds} + \frac{d\varepsilon_4}{ds}\cdot\varphi\right\} = 0. \end{aligned} \tag{23}$$

Since l, m, n are arbitrary, it can easily be shown that

$$\Gamma^1_{22} = 0,\ \Gamma^1_{44} = 0,\ \Gamma^1_{24} = 0,\ \Gamma^1_{12} = 0,\ \Gamma^1_{14} = 0, \tag{24}$$

$$\varphi = c,\ \frac{d\varphi}{ds} = 0, \tag{25}$$

where c is a constant. Therefore the equation (23) becomes

$$\frac{d^2\varepsilon_1}{ds^2}\cdot\varphi + \Gamma^1_{11}\left(\frac{d\varepsilon_1}{ds}\right)^2\cdot\varphi^2 = 0, \tag{26}$$

which gives

$$\Gamma^1_{11} = \frac{d}{ds}\left(1\bigg/\frac{dx}{ds}\right). \tag{27}$$

Similarly it can be shown that

$$\Gamma^2_{22} = \frac{d}{ds}\left(1\bigg/\frac{dy}{ds}\right), \tag{28}$$

$$\Gamma^4_{44} = \frac{d}{ds}\left(1\bigg/\frac{dt}{ds}\right). \tag{29}$$

Similarly the other expressions for $\Gamma^\alpha_{\beta\gamma}$, s can be shown to be equal to zero.

As a particular case if we define ε_1 as

$$e^{\varepsilon_1 s + x_0 + l_0} = e^x = s \pm l, \tag{30}$$

then differentiating,

$$\frac{dx}{ds} = \frac{1}{s \pm l}. \tag{31}$$

Therefore, $$\Gamma^1_{11} = 1. \tag{32}$$

Similarly if $$e^y = s \pm m;\ e^t = s \pm n, \tag{33}$$

then $$\Gamma^2_{22} = 1,\ \Gamma^4_{44} = 1. \tag{34}$$

It has been shown by Eisenhart (I927) that the necessary and sufficient condition that a space with an assigned symmetric connection be Riemannian is that the equations

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - g_{hj}\Gamma^h_{ik} - g_{ih}\Gamma^h_{jk} = 0 \tag{35}$$

admit a solution. In this case, the equaton turns out to be

$$g_{11,1} = \frac{\partial g_{11}}{\partial x} - 2g_{11} = 0. \tag{36}$$

A particular solution of this is

$$g_{11} = e^{2x}. \tag{37}$$

Similarly, other particular solutions can be shown to be given by

$$g_{12} = e^{x+y},\ g_{14} = e^{x+t},\ g_{24} = e^{y+t} \text{ etc.} \tag{38}$$

Thus the metric is given by

$$ds^2 = e^{2x}dx^2 + e^{2y}dy^2 + e^{2t}dt^2 + 2e^{x+y}dxdy + 2e^{x+t}dxdt + 2e^{y+t}dydt. \tag{39}$$

Case (iv). Let the equations of the path be of the form

$$\frac{1}{r} = \frac{m}{h^2}(1 + e\cos\theta), \tag{40}$$

$$t = k\varphi(\theta), \tag{41}$$

$$\frac{d\theta}{ds} = f(r, \theta, t), \tag{42}$$

where m, h, e and k are arbitrary constants. Then the different differential coefficients are given by

$$\frac{dt}{ds} = k\varphi(\theta)\cdot f(r, \theta, t), \tag{43}$$

$$\frac{dr}{ds} = \frac{h^2}{m}\cdot e\cdot\frac{\sin\theta}{(1+e\cos\theta)^2} f(r, \theta, t), \tag{44}$$

$$\frac{d^2\theta}{ds^2} = f(r, \theta, t)\left[\frac{\partial f}{\partial r}\cdot\frac{eh^2}{m}\frac{\sin\theta}{(1+e\cos\theta)^2}+\frac{\partial f}{\partial\theta}+\frac{\partial f}{\partial t}\cdot k\varphi'(\theta)\right], \tag{45}$$

$$\frac{d^2t}{ds^2} = k\varphi''(\theta)\cdot f^2+k\varphi'(\theta)\cdot\frac{d^2\theta}{ds^2}, \tag{46}$$

$$\frac{d^2r}{ds^2} = \frac{eh^2}{m(1+e\cos\theta)^2}\left[\frac{(\cos\theta+e+e\sin^2\theta)}{(1+e\cos\theta)}\cdot f^2+\sin\theta\,\frac{d^2\theta}{ds^2}\right]. \tag{47}$$

Substituting these in an equatisn of (1), the following is obtained:

$$f\left[\frac{\partial f}{\partial r}\cdot\frac{eh^2}{m}\cdot\frac{\sin\theta}{(1+e\cos\theta)^2}+\frac{\partial f}{\partial\theta}+\frac{\partial f}{\partial t}\cdot k\varphi'(\theta)\right]+\Gamma^2_{11}\frac{e^2h^4}{m^2}\,\frac{\sin^2\theta}{(1+e\cos\theta)^4}\cdot f^2$$
$$+\Gamma^2_{22}f^2+\Gamma^2_{44}\varphi'^2k^2f^2+2\Gamma^2_{24}kf^2\varphi'+2\Gamma^2_{14}\,\frac{eh^2}{m}\cdot\frac{\sin\theta}{(1+e\cos\theta)^2}\cdot f^2k\varphi'$$
$$+2\Gamma^2_{12}\frac{eh^2}{m}\cdot\frac{\sin\theta}{(1+e\cos\theta)^2}\cdot f^2 = 0. \tag{48}$$

Since m, h, e, k are arbitrary, it can be shown that

$$\Gamma^2_{44} = 0,\quad \Gamma^2_{11} = 0,\quad \Gamma^2_{14} = 0, \tag{49}$$

$$2\Gamma^2_{12}f+\frac{\partial f}{\partial r} = 0, \tag{50}$$

$$2\Gamma^2_{24}f+\frac{\partial f}{\partial t} = 0, \tag{51}$$

$$\Gamma^2_{22}f+\frac{\partial f}{\partial\theta} = 0. \tag{52}$$

By defining f differently, different expressions for $\Gamma^2_{\beta\gamma}$ in terms of the coordinates, are obtained. Similarly it can be shown that the other equations of the path give the following:

$$\Gamma^1_{44} = 0,\ \Gamma^1_{22} = 0,\ \Gamma^1_{24} = 0, \tag{53}$$

$$2\Gamma^1_{14}f+\frac{\partial f}{\partial t} = 0, \tag{54}$$

$$2\Gamma^1_{12}f\sin\theta+\frac{\partial f}{\partial\theta}\cdot\sin\theta+f\,\frac{\cos\theta+e+e\sin^2\theta}{1+e\cos\theta} = 0, \tag{55}$$

$$\Gamma^1_{11}f+\frac{\partial f}{\partial r}=0; \tag{56}$$

$$\Gamma^4_{11}=0,\ \ \Gamma^4_{22}=0,\ \ \Gamma^4_{12}=0, \tag{57}$$

$$2\Gamma^4_{14}f+\frac{\partial f}{\partial r}=0, \tag{58}$$

$$2\Gamma^4_{24}f\varphi'+\frac{\partial f}{\partial\theta}\cdot\varphi'+f\varphi''=0, \tag{59}$$

$$\Gamma^4_{44}f+\frac{\partial f}{\partial t}=0. \tag{60}$$

So it is clear that by assuming different forms for f, different sets of $\Gamma^\alpha_{\beta\gamma}$'s are obtained though the form of the path remains intact all the while. This means that the same form of the path is compatible with different affine connections. It has been shown by Eisenhart (1927) that the paths are the same for two symmetric connections whose coefficients are in the relation

$$\overline{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_j\psi_k+\delta^i_k\psi_j \tag{61}$$

where ψ_i is an arbitrary covariant vector.

If there is a space with the symmetric connection $\overline{\Gamma}^i_{jk}$ then it is not always possible to express it as $\Gamma^i_{jk}+\delta^i_j\psi_k+\delta^i_k\psi_j$. So when $\overline{\Gamma}^i_{jk}$ belongs to a non-Riemannian space, it is not always possible to find the corresponding Riemannian space with the same path. This means all paths are not compatible with a metric hypothesis. Also, it is obvious that any path existing in a Riemannian space, must also exist in a space which has no metric.

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Summary—Four different arbitrary forms of paths are taken in a non-Riemannian manifold and in each case affine-connections are searched for. In the first case, it is found that the values of $\Gamma^\alpha_{\beta\gamma}$'s are zero. In the second case it is found that $\Gamma^\alpha_{\beta\gamma}$'s cannot be expressed in terms of the coordinates which physically means that there exists no space with a symmetric connection such that equations of the path can be as taken in this case. In the third case, $\Gamma^\alpha_{\beta\gamma}$'s are shown expressible in terms of coordinates and their values under particular conditions are determined. Applying the necessary and

sufficient condition for a space with an assigned symmetric connection to be Riemannian, a metric is evolved as a particular solution. In the last case dealing with the fourth form of the path, different sets of $\Gamma^{\alpha}_{\beta\gamma}$'s under different conditions, are obtained. The physical significance of this result is that the same form of the path is compatible with different affine-connections.

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A THEOREM ON OPERATIONAL CALCULUS

By

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1. The object of this paper is to establish the following theorem:

The function $f(x)$ *determined by*

$$f(p) = p\int_0^\infty e^{-ps}\varphi(s)\,ds \tag{1. 1}$$

satisfies the integral equation

$$f(p+b) = \mu(p+b)\int_0^\infty \frac{x^\lambda}{x+b}J_\lambda^\mu\{(p-c)\,x^\mu\}\,f(x+b)\,dx \tag{1. 2}$$

when the functional relation

$$\varphi(s) = s^{(\lambda/\mu)-1}e^{(c+b)s}e^{-bs^{-1/\mu}}\varphi(s^{-1/\mu}) \tag{1. 3}$$

holds.

This theorem is valid for

$$(i)\quad \varphi(x) = 0(x^{-1+\epsilon})\quad \textit{when } x \textit{ is small}$$

$$(ii)\quad \varphi(x) = 0(x^\eta e^{kx})\quad \textit{when } x \textit{ is large}$$

provided $R(p) > 0;\ R(\lambda) > -1;\ R(\eta) > -1;\ 0 < \mu < 1;$

$R(p-k) > 0;\ R(b-k) \geq 0;\ R(p-c) > 0$

The theorem also holds for

$$(i)\quad R(b-k) = 0 \quad \textit{provided}\quad R(\lambda-\eta) > 0$$

$$(ii)\quad \mu = 1 \qquad \textit{provided}\quad R(\lambda-2\eta) < \tfrac{5}{2}$$

The function $J_\lambda^\mu(x)$ (*Wright, 1935*) *is defined by*

$$J_\lambda^\mu(x) = \sum_{r=0}^\infty \frac{(-x)^r}{\underline{\lfloor r}\ \Gamma(1+\lambda+\mu r)} \qquad (\mu > 0)$$

Proof: We know that (Mclachlan) if

$$\varphi(s) \supset f(p),$$

then

$$e^{-bs}\varphi(s) \supset \frac{p}{p+b}f(p+b);\quad R(p+b-k) > 0 \tag{1. 4}$$

Also, termwise interpretation gives

$$s^\lambda J_\lambda^\mu(a\,s^\mu) \supset p^{-\lambda}\exp(-a\,p^{-\mu});\quad \mu > 0,\ R(\lambda) > -1,\ R(p) > 0 \tag{1. 5}$$

Applying Parseval Goldstein Theorem (Goldstein, 1932) to the relations (1.4) and (1. 5) we get

$$\int_0^\infty t^{-\lambda-1}\exp(-at^{-\mu})\, e^{-bt}\varphi(t)\, dt = \int_0^\infty \frac{x^\lambda}{x+b} J^\mu_\lambda(ax^\mu)\, f(x+b)\, dx$$

provided $R(p) > 0$; $R(\eta) > -1$; $R(\lambda) > -1$; $R(p-k) > 0$; $R(b-k) > 0$; $\mu < 1$ which on substituting $s = t^{-\mu}$ and $a = p$ may be written as

$$p\int_0^\infty s^{(\lambda/\mu)-1} e^{-ps} e^{-bs^{-1/\mu}} \varphi(s^{-1/\mu})\, ds = \mu p \int_0^\infty \frac{x^\lambda}{x+b} J^\mu_\lambda(px^\mu)\, f(x+b)\, dx$$

whence

$$s^{(\lambda/\mu)-1} e^{-bs^{-1/\mu}} \varphi(s^{-1/\mu}) \supset \mu p \int_0^\infty \frac{x^\lambda}{x+b} J^\mu_\lambda(px^\mu)\, f(x+b)\, dx$$

Applying the rule (1. 4) to this relation, we see that

$$s^{(\lambda/\mu)-1} e^{cs} e^{-bs^{-1/\mu}} \varphi(s^{-1/\mu}) \dot{\supset} \mu p \int_0^\infty \frac{x^\lambda}{x+b} J^\mu_\lambda\{(p-c)x^\mu\} f(x+b)\, dx$$

provided $R(\lambda) > -1$; $R(p-c) > 0$

If now $\varphi(s)$ satisfies the functional relation

$$e^{-bs}\varphi(s) = s^{(\lambda/\mu)-1} e^{cs} e^{-bs^{-1/\mu}} \varphi(s^{-1/\mu})$$

or

$$\varphi(s) = s^{(\lambda/\mu)-1} e^{(c+b)s} e^{-bs^{-1/\mu}} \varphi(s^{-1/\mu})$$

then, by Lerch's Theorem (Lerch, 1903), it follows that

$$f(p+b) = \mu(p+b) \int_0^\infty \frac{x^\lambda}{x+b} J^\mu_\lambda\{(p-c)x^\mu\}\, f(x+b)\, dx$$

Example:—Let $\varphi(s) = e^{bs} s^{\nu-1}$ and $c = 0$, then the functional relation (1. 3) is satisfied provided $\mu = \dfrac{\lambda-\nu+1}{\nu}$

Therefore

$$f(p) = p\int_0^\infty e^{-ps} e^{bs} s^{\nu-1} ds$$

$$R(p-b) > 0;\ R(\nu) \gtrsim 0$$

$$= \Gamma(\nu)\frac{p}{(p-b)^\nu}$$ (Magnus & Oberhettinger,)

satisfies the integral equation (1. 2) provided

$$R(p-b) \geq 0;\ R(2\nu) > R(\lambda+1) > R(\nu) > 0$$

2. If $f(x)$ is the function determined by (1. 1) where $\varphi(s)$ satisfies the functional relation

$$\varphi(s) = s^{\lambda-1} e^{b(s-1/s)} \varphi(1/s) \tag{2. 1}$$

then, the function

$$\frac{x^{\lambda+\frac{1}{2}}}{x^2+2b}f\left(\frac{x^2}{2}+b\right) \quad \text{is} \quad R_\lambda{}^*$$

This theorem is valid for

$$(i) \quad \varphi(x) = 0(x^{-1+\epsilon}) \qquad \text{when } x \text{ is small}$$

$$(ii) \quad \varphi(x) = 0(x^{\eta}e^{kx}) \qquad \text{when } x \text{ is large}$$

provided

$$R(p) > 0; \; R(\eta) > -1; \; R(\lambda) > -1$$

$$R(\lambda-2\eta) < \tfrac{5}{2}; \; R(p-k) > 0; \; R(b-k) > 0$$

For on putting $\mu = 1$ and $c = 0$, (1. 3) reduces to (2. 1) and (1. 2) takes the form

$$f(p+b) = (p+b)\int_0^\infty \frac{x^\lambda}{x+b}J_\lambda^1(px)\, f(x+b)\, dx$$

or

$$\frac{p^{\lambda/2}}{p+b}f(p+b) = \int_0^\infty \frac{x^{\lambda/2}}{x+b}J_\lambda\{2\surd(px)\}\, f(x+b)\, dx$$

which can be written in the form

$$\frac{y^{\lambda+\frac{1}{2}}}{y^2+2b}\, f(\tfrac{1}{2}y^2+b) = \int_0^\infty \surd(xy)\, J_\lambda(xy)\frac{x^{\lambda+\frac{1}{2}}}{x^2+2b}\, f(\tfrac{1}{2}x^2+b)\, dx$$

showing that

$$\frac{x^{\lambda+\frac{1}{2}}}{x^2+2b}\, f(\tfrac{1}{2}x^2+b) \text{ is } R^\lambda$$

For $b = 0$, this gives that

$$x^{\lambda-3/2}f(\tfrac{1}{2}x^2) \text{ is } R_\lambda$$

provided the functional relation

$$\varphi(s) = s^{\lambda-1}\varphi(1/s)$$

is satisfied.

This is the result due to Gupta (1943)

As an illustration, we see that

$$\varphi(s) = s^{\frac{1}{2}\lambda-\frac{1}{4}}\, e^{-(a-b)s-a/s}$$

satisfies the functional relation (2. 1) and then (Mclachlan and Humbert, 1950)

$$f(p) = p\int_0^\infty e^{-ps}s^{\frac{1}{2}\lambda-\frac{1}{4}}e^{-(a-b)s-a/s}\, ds$$

$$= 2p\left(\frac{p+a-b}{a}\right)^{-\frac{1}{2}\lambda-\frac{1}{4}}k_{-\frac{1}{2}\lambda-\frac{1}{2}}\,[2\surd\{a(p+a-b)\}]$$

We, therefore, get the known result that (Watson, 1944)

$$x^{\lambda+\frac{1}{2}}(x^2+2a)^{-\frac{1}{2}\lambda-\frac{1}{4}}k_{-\frac{1}{2}\lambda-\frac{1}{2}}[\surd\{2a(x^2+2a)\}] \text{ is } R_\lambda$$

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[*] A function is said to be R_ν, if it is self-reciprocal the Hankel-Transform of order ν.

provided $$R(\lambda) > -1;.$$

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ON BOOLEAN NARINGS

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1. A system in which all the axioms for a ring are satisfied except the associative law of multiplication which may even be replaced by a weaker law, is called a *not necessarily associative ring* or more simply a *naring* (Brown and McCoy, 1950). A Boolean naring is a naring in which every element is idemponent multiplicatively. The object of this note is to find out how far a Boolean naring behave like a Boolean ring. It is well known that a Boolean ring can be distorted into a Boolean algebra and conversely. The associative law leads to the important fact that inclusion in the Boolean algebra is transitive. It has been shown here that a Boolean naring can be distorted into a system in which the inclusion is reflexive, assymmetric but not transitive. Such a system will be called a *trellis.* Here the weak associative law $(aa)b = a(ab)$ which has been called the (12)-alternative law, plays an interesting role. It has been proved here that if a Boolean naring is (12)-alternative, then it can be distorted into a Boolean algebra, consequently it follows that an (12)-alternative Boolean naring is a Boolean ring.

2. The naring in which the associator $(a, b, c) = a(bc) - (ab)c \neq 0$ is an alternative function of a, b, c was first discussed Zorn and Moufang. A transposition of two elements, say $a \longleftrightarrow b$ changes the sign of the associator (a, b, c). Here one takes the characteristic of the naring to be different from 2. The associator (a, b, c) remains therefore invariant under any even permutation of a, b, c. One may generalise this notion further and consider narings in which the associator (a, b, c) changes its sign only for a particular permutation, say by $a \longleftrightarrow b$. Such a naring may be called an (12)-alternative naring. Similarly one can define (13) or (23)-alternative narings.

In a naring of characteristic $\neq 2$, the commutative law implies the (13)-alternative law For,

$$(a, b, c) = a(bc) - (ab)c = -[c(ba) - (cb)a] = -(c, b, a)$$

This shows the importance of semialternative laws.

In an (12)-alternative naring, $(a, b, c) = -(b, a, c)$. Putting $a = b$, one gets that $(a, a, c) = -(a, a, c)$. As the characteristic is $\neq 2$, this implies that $(a, a, c) = 0$ *ie.* $a(ac) = (aa)c$. Conversely if this condition holds, the naring is (12)-alternative; for under this supposition, $(a+b,\ a+b,\ c) = 0$.

Hence $$(a, a, c) + (a, b, c) + (b, a, c) + (b, b, c) = 0$$

ie. $$(a, b, c) = -(b, a, c).$$

Thus the necessary and sufficient condition that a naring may be (12)-alternative is that $(a, a, b) = 0$. Similarly the necessary and sufficient conditions that a naring may be (13)-and (23)-alternative are $(a, b, a) = 0$ and $(b, a, a) = 0$ respectively.

If a naring is (13)-and (12)-alternative, then it is also (23)-alternative, For, by the (13)-alternative law,

$$(a, a, b) = -(b, a, a).$$

But, by the (12)-alternative law, $(a, a, b) = 0$ and hence $(b, a, a) = 0$. As every permutation of three elements a, b, c can be expressed as a product of these permutations, the associator (a, b, c) is an alternative function of a, b, c. Thus the naring is fully alternative. One can apply similar considerations to other cases and get that a naring which is semialternative in two ways, is fully alternative. Further, as the commutativety implies the (13)-alternative law, a commutative (12)-alternative naring is fully alternative.

If the characteristic of the naring is 2, the above results are obtained by a slight modification of the definition. Here one defines that a naring is (12)-alternative if $(a, a, b) = 0$. Similarly it is (13)-and (23)-alternative according as $(a, b, a) = 0$ and $(b, a, a) = 0$ respectively, Here $(a, a, b) = 0$ implies that $(a, b, c) = (b, a, c)$ so that the (12)-alternative condition is really the condition for (12)-symmetry. In spite of this fact, we shall call this condition as the alternative condition for the sake of uniformity.

If a naring is commutative and of characteristic 2, then, as $(a, b, c) = a(bc)+(ab)c$, the commutativity implies that $(a, b, c) = (c, b, a)$ which shows that the associator is (13)-symmetric. But this does not not imply $(a, b, a) = 0$, so that the commutativity does not imply here the (13)-alternative condition. But as the commutativity implies (13)-symmetry, $(a, a, b) = (b, a, a)$. Thus if $(a, a, b) = 0$, then $(b, a, a) = 0$. Hence a commutative (12)-alternative naring is also (23)-alternative. Further by this (23)-alternative condition, $(a, b, a) = (a, a, b) = 0$. Thus the naring is also (23)-alternative. This shows that a commutative (12)-alternative naring of characteristic 2 is also fully alternative.

Moufang laws can easily be proved for naring of characteristic 2. It is easy to see that the four identity

$$(ab, c, d)+(a, b, cd)+(a, bc, d) = a(b, c, d)+(a, b, c)d$$

also holds here. Putting $a = b$ in this relation and using (12)-alternative condition, one gets

$$(a^2, c, d)+(a, ac, d) = a(a, c, d).$$

Interchanging c with d in this relation and adding to it, the relation so obtained one gets $(a, ac, d)+(a, ad, c) = 0$ which, on simplification, gives that $a(c(ad)) = ((ac)a)d$. This is one of Moufang laws and the other law $(ad)(ca) = a((dc)a)$ can be obained similarly.

Semialternative narings have another interesting property. Denoting

$$\begin{aligned}[a, b] &= ab-ba \quad \text{if the characteristic} \neq 2\\ &= ab+ba \quad \text{if the characteristic} = 2.\end{aligned}$$

one proves easily,

$$[a,[b, c]]+[b,[c, a]]+[c,[a, b]] = \{a, b, c\}-\{b, a, c\}$$

where $\{a, b, c\} = (a, b, c)+(b, c, a)+(c, a, b) = [a, bc]+[b, ca]+[c, ab]$ and $\{c, b, a\}$ is obtained from $\{a, b, c\}$ by replacing a by c. $\{a, b, c\}$ is independent of the cyclic permutation and hence may called a *circulant*. For a Lie naring, the left side vanishes and hence $\{a, b, c\} = \{b, a, c\}$. Also if a naring is (12)-alternative, $\{a, b, c\} = -\{c, b, a\}$, hence

$$[a,[b, c]] + [b,[a, c]] + [c,[a, b]] = 2\{a, b, c\}$$

This result is also true for a naring of characteristic 2. Thus if an (12)-alternative naring is a Lie naring, the circulant must vanish; but an (12)-alternative naring is always a Lie naring if the characteristic $= 2$

3 A naring whose elements are multiplicatively idempotent will be called a Boolean naring. As the distributive law holds,

$$(x+y)^2 = x^2+xy+yx+y^2$$

so that idempotency gives $xy+yx = 0$ *i.e.* $xy = -yx$.
Now if $x = y$, one gets $2x^2 = 0$ *i.e.* $2x = 0$ for every x, so that the characteristic is 2. Consequently $xy = -yx = yx$. Thus the multiplication is idempotent and commutative, though nonassociative.

Define in this naring, an inclusion relation by the condition, $x \geq y$ if and only if $xy = y$. Then as $x^2 = x$, $x \geq x$ and as the commutative law holds, $x \geq y$, $y \geq x$ implies that $x = y$. But, as the associative law is not true, the transitive law for the inclusion is not necessarily true.

Now suppose, for the present, that the naring has a unique both sided identity 1. Then

$$(1-y)(1-x) = 1-y-x+xy = 1-y-x+y = 1-x$$

Putting $\quad x' = 1-x,\ y' = 1-y,$ one gets $x'y' = x'$.

Thus if $\quad x \geq y,\ y' \geq x'$. Also $x'' = 1-(1-x) = x$.

Next distort the sum by the rule $\quad x{*}y = (x'y')'$.

Then $\quad (x{*}y)' = x'y'$

and $\quad (xy)' = (x''y'')' = ((x'{*}y')')' = x'{*}y'$.

Hence $x \rightarrow x'$ is a dual automorphism inverting the inclusion. Further the new addition is expressible in terms of the old addition and the multiplication. Indeed

$$x{*}y = (x'y')' = 1-x'y' = 1-(1-x)(1-y) = x+y-xy$$

Idempotency and commutativity are easily proved for this addition. But the associative law is not true for this addition. Indeed

$$(x{*}y){*}z = x+y+z-xy-xz-yz+(xy)z$$

and hence the associative law of this addition involves that of multiplication.

Again, if $x{*}y = x$, then $x+y-xy = x$; hence $xy = y$. This shows that the inclusion for this new addition is the converse of the inclusion for the multiplication. Also the absorptive and the distributive laws do not hold. For,

$$x{*}(yx) = x+yx-x(yx) \neq x \text{ as } x(yx) \neq yx$$

$$x(y{*}z) = xy+xz-x(yz) \neq xy+zz-(xy)(xz) = xy{*}xz.$$

Further, $$x_*x' = x+(1-x)-x(1-x) = 1$$

$$xx' = x(1-x) = x-x = 0$$

so that x' is the complement of x.

Also as $$x1 = x \qquad x_*1 = 1$$

$$x0 = 0 \qquad x_*0 = x$$

A system in which a reflexive and assymmetric inclusion is defined, may be called a *trellis*. Thus one gets that a Boolean naring can be distorted into a complemented trellis with 0 and 1 as the least and the greatest element.

Now let the naring be (12)-alternative. Then

$$x(xy) = (xx)y = xy$$

and hence $$x_*yx = x+yx-x(xy) = x+yx-xy = x.$$

$$x(x_*y) = x(x+y-xy) = x+xy-x(xy) = x.$$

Further as the Boolean naring is commutative, it is fully alternative if it is (12)-alternative. Consequently in an (12)-alternative Boolean naring Moufang laws hold. Thus

$$(xy)(xz) = (xy)(zx) = x((yz)x) = (xx)(yz) = x(yz)$$

and so
$$\begin{aligned} xy_*xz &= xy+xz-(xy)(xz) \\ &= xy+xz-x(yz) \\ &= x[y+z-yz] \\ &= x(y_*z). \end{aligned}$$

In a Boolean algebra, the associative law of multiplication holds. As the multiplication in the Boolean algebra is same as in the Boolean (12)-alternative naring, the associative law of multiplication holds in this naring also Indeed the Boolean algebra may be distorted back into a ring by the distortion $a \oplus b = ab'_*a'b$. This addition $a \oplus b$ is same as the old addition $a+b$. For,

$$\begin{aligned} a \oplus b &= ab'_*a'b \\ &= a(1-b)+(1-a)b-[a(1-b)][(1-a)b] \\ &= (a-ab)+(b-ab)-[ab-(ab)b-a(ab)+(ab)(ab)] \\ &= a+b. \end{aligned}$$

Thus by distorting an (12)-alternative Boolean naring into a Boolean algebra which is again distorted back into a ring, one gets that the associative law of multiplication in a Boolean ring is a consequence of (12)-alternative law $a(ab) = (aa)b$.

If the naring does not contain a unique both sided identity then one considers the relative complements of elements, instead of complements. If a is arbitrary element and $0 \leq x \leq a$, then putting $x = a-x$,

$$x'' = a-(a-x) = x$$

$$xx' = x(a-x) = ax-x^2 = 0 \text{ as } ax = x,\ x^2 = x.$$

Also one can distort the sum by the rule

$$x{*}y = (x'y')'$$

so that $x{*}y = x+y-xy$ as before and the new sum is independent of a The trellis obtained by distortion will be absorptive and distributive Consequently, here also the trellis will reduce to a distributive lattice.

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INVERSION OF INCOMPLETE BLOCK DESIGNS

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Abstract. This paper deals with some problems regarding inversion of incomplete block designs (not necessarily balanced in the sense of Yates, [1936] or partially balanced in the sense of Bose and Nair, [1939], and Nair and Rao, [1942]), the object being to study the correspondence between any two of affine resolvible, balanced and partially balanced incomplete block designs. Some interesting results have been obtained and it has been possible to show that there is a one-to-one correspondence between affine resolvable designs (unbalanced, balanced and partially balanced) and a certain class of GD designs (Bose, 1951; Bose and Shimamoto, 1952) but in general there is no one-to-one correspondence between balanced and partially balanced incomplete block designs though in certain general cases there is one-to-one correspondence.

1. Introduction.

In an earlier communication (which is awaiting publication in Sankhyā the author has proved that partially balanced incomplete block (p. b. i. b.) designs (Bose and Nair, 1939; Nair and Rao, 1942) always result from the inversion of certain classes of balanced incomplete block (b. i. b.) designs (Yates, 1936), namely that p. b. i. b. designs result from the inversion of

(1) a b. i. b. design with parameters $v, b, r, k, \lambda = 1$ $(v \neq b)$,

(2) a b. i. b. design with parameters $v' = v'' - k''$, $b' = b'' - 1$, $r' = r''$, $k' = k'' - 2$, $\lambda' = 2$ (obtained by block-section from the b. i. b. design $v'' = b''$, $r'' = k''$, $\lambda'' = 2$),

(3) an affine resolvable b. i. b. design with parameters $v = n^2(n-1)\,t + n^2$, $b = n(n^2 t + n + 1)$, $r = n^2 t + n + 1$, $k = n(n-1)t + n$, $\lambda = nt + 1$, and

(4) a b. i. b. design

$$v,\ b = {}^{v}C_{k},\ \ r = {}^{v-1}C_{k-1},\ \ k,\ \lambda = {}^{v-2}C_{k-2}\ \ (k \neq v-1)$$

(in which all the combinations have been completely written down). Further it has been shown that if a b. i. b. design on inversion gives either a b. i. b. design or a p. b. i. b. design, then by inverting a b. i. b. design which is n times the first we shall always get a p. b. i. b. design which will be the same as the p. b. i. b. design which can be obtained by replacing each vatiety of the inverted original b. i. b. design by a group of n varieties; also if a b. i. b. design on inversion becomes a p. b. i. b. design, then the complementary b. i. b. design also on inversion will become a p. b. i. b. design which will be complementary to the p. b. i. b. design obtained by inverting the original b. i. b. design.

Recently it has come to the notice of the author that Plackett and Burman (1946), while discussing multifactorial designs, observed the duality relation (3) between affine resolvable b. i. b. and p. b. i. b. designs. It may be noted that the relations (1) and (2) have also been obtained meanwhile by Shrikhande (1952) in different forms. The object of the present paper is to deal with the problems of inversion of incomplete block designs in general (*i. e.* not necessarily inversion of b. i. b. designs).

2. Inversion of Linked Block Designs.

Suppose there is an incomplete block arrangement of v varieties in b blocks of k units each, each variety being replicated r times. The method of inversion consists of taking the varieties of this arrangement as blocks and the blocks as varieties. In this way we get a new arrangement which will be called the inverted design of b. k units of b varieties, each variety being replicated k times, into v blocks of size r each. If in the original arrangement B_i and B_j be any two blocks having exactly λ varieties in common, then in the inverted design the varieties B_i and B_j will occur together exactly in λ blocks. From this it is clear that if λ is independent of i and j, then the inverted arrangement must be a b. i. b. design. If λ is not independent of i and j, then the inverted arrangement cannot be a b. i. b. design. From these discussions it follows at once that all the linked block designs of Youden (1951) on inversion will turn into b. i. b. designs. On the other hand p. b. i. b. designs obtained by inverting b. i. b. designs should always be 'λ times linked block designs'. Consequently *p. b. i. b. designs obtained by inverting b. i b. designs can never be resolvable.*

Fisher (1940) has proved that 'if a balanced incomplete block design with parameters v, b, r, k, λ has the property that any two blocks of the design have the same number of varieties in common, then the design is symmetrical *i e.* $b = v$; conversely if the design is symmetrical *i.e.* if for the design the condition $b = x$ holds, then any two blocks have exactly the same number of varieties in common *viz.* λ'. Utilising this we can now not only state that "*only symmetrical b. i. b. designs and no other b. i. b. design on inversion will remain b. i. b. designs,*" and that "*a p. b. i. b. design for which $b = v$ can not be a 'λ times linked block design' and on inversion can never turn into a b. i. b. design,*" but we can also draw the following two important conclusions:

(a) For Youden's linked block designs the inequality $v > b$ must necessarily hold as otherwise by inverting those designs we shall get b. i. b. designs for which Fisher's inequality $b \geq v$ should not hold.

(b) Any incomplete block design arrangement with $b = v$, $r = k$, which is λ times linked (*i. e.* of which any two blocks have exactly λ varieties in common) must be a b. i. b. design. In this connection it is worth noting that any incomplete arrangement with v, b, r, k, $(v \neq b)$, which is either singly linked or doubly linked with $r = k-2$ can not be other than a p. b. i. b. design with two associate classes. This is due to the results (1) and (2) mentioned in the introduction.

It may also be noted that if an arrangement (which need not be a b. i. b. or p. b. i. b. design) of $v. r.$ units of v varieties, there being r units of each variety but two units of the same variety not occuring in a block, into b blocks of size k be such that

(i) any block B_i of the arrangement has exactly j varieties in common with f_j blocks ($j = 0, 1, 2, \ldots, n$, $n \leq k$), f_j being independent of i, and

(ii) there are exactly $p^i_{jj'}$ blocks, each of which has exactly j varieties in common with ${}_iB_p$ and j' varieties in common with ${}_iB_q$ (for any i and all p, q), where ${}_iB_p$ and ${}_iB_q$ are any two blocks of the arrangement, having just i treatments in common, then the inverted arrangement must be a p. b. i. b. design.

The following example of inverted p. b. i. b. design will reveal certain interesting relations regarding inversion:

Example (2. 1): Let us consider the following solution of the p. b. i. b. design with parameters

$$\left.\begin{array}{l}
v = 8, \quad b = 6, \quad r = 3, \quad k = 4 \\
n_1 = 3, \quad n_2 = 3, \quad n_3 = 1 \\
\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = 0 \\
p^1_{jj'} = \begin{Vmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix}, \quad p^2_{jj'} = \begin{Vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{Vmatrix}, \quad p^3_{jj'} = \begin{Vmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}
\end{array}\right\} \tag{2.01}$$

$$\left.\begin{array}{l}
B_1(1, 2, 3, 4), \quad B_2(1, 2, 5, 6), \quad B_3(1, 4, 5, 8) \\
B_4(5, 6, 7, 8), \quad B_5(3, 4, 7, 8), \quad B_6(2, 3, 6, 7)
\end{array}\right\} \tag{2.02}$$

(Bose and Nair, 1939, page 353 Ex. (IV))

Inverting this design we get

$$\left.\begin{array}{l}
1.\ (B_1\ B_2\ B_3), \quad 2.\ (B_1\ B_2\ B_6), \quad 3.\ (B_1\ B_5\ B_6), \quad 4.\ (B_1\ B_3\ B_5) \\
7.\ (B_4\ B_5\ B_6), \quad 8.\ (B_3\ B_4\ B_5), \quad 5.\ (B_2\ B_3\ B_4), \quad 6.\ (B_2\ B_4\ B_6)
\end{array}\right\} \tag{2.03}$$

which is a solution of the p. b. i. b. design with parameters

$$\left.\begin{array}{l}
v = 6, \quad b = 8, \quad k = 3, \quad r = 4, \\
n_1 = 4, \quad n_2 = 1, \quad \lambda_1 = 2, \quad \lambda_2 = 0, \\
p^1_{jj'} = \begin{Vmatrix} 2 & 1 \\ 1 & 0 \end{Vmatrix}, \quad p^2_{jj'} = \begin{Vmatrix} 4 & 0 \\ 0 & 0 \end{Vmatrix}
\end{array}\right\} \tag{2.04}$$

(Bose and Nair, 1939, page 354, (v))

It is interesting to note that the solution of the first design is affine resolvable whereas that of the second one is simple resolvable. Moreover the last mentioned design, namely (2.04) can be *parametrically reduced* by cancelling the common factor 2 between b, r, λ_1 and λ_2.

The parameters of the reduced design are

$$\left.\begin{array}{l}
v = 6, \quad b = 4, \quad k = 3, \quad r = 2 \\
n_1 = 4, \quad n_2 = 1, \quad \lambda_1 = 1, \quad \lambda_2 = 0 \\
p^1_{jj'} = \begin{Vmatrix} 2 & 1 \\ 1 & 0 \end{Vmatrix}, \quad p^2_{jj'} = \begin{Vmatrix} 4 & 0 \\ 0 & 0 \end{Vmatrix}
\end{array}\right\} \tag{2.05}$$

For (2.05) we get the following solution, by considering the scratched faces of the octahedron (Fig. 1) as blocks and the points as varieties

$$(1, 2, 3),\ (3, 4, 5),\ (5, 1, 6),\ (6, 4, 2) \tag{2.06}$$

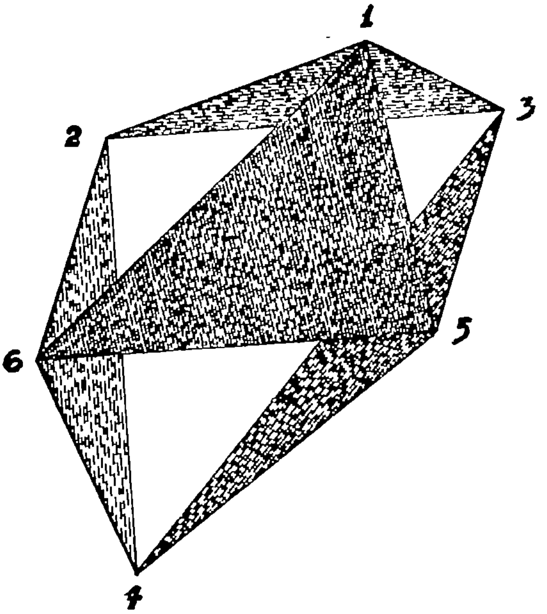

FIG. 1

Similarly by considering the blank faces we get another solution for (2.05) as

$$(4, 5, 6),\ (6, 1, 2),\ (2, 4, 3),\ (3, 5, 1) \tag{2.07}$$

These two solutions together is the solution which Bose and Nair have given for the p. b. i. b design (2.04). By duplicating either (2.06) or (2.07) we can obtain another solution for (2.04) which unlike (2.03) is non-resolvable. Now as the solution (2.03) has been obtained by inverting the solution (2.02), the solution of (2.03) on inversion will turn into (2.02); but what about the solution of (2.04) obtained by duplicating the solution (2.06)? To answer this question let us note that the solution (2.06) or (2.07) is a 'singly linked block' design in the sense of Youden. That is, any two blocks of (2.06) or (2.07) have just one variety in common. Therefore, when inverted both of (2.06) and (2.07) will turn into a b. i. b. design with parameters $v = 4$, $b = 6$, $r = 3$, $k=2$, $\lambda=1$, and twice of (2.06) or (2.07) will turn into a p. b. i. b. design with parameters

$$\left.\begin{array}{l} v = 8, \quad b = 6, \quad r = 3, \quad k = 4 \\[6pt] n_1 = 1, \ n_2 = 6, \ \lambda_1 = 3, \ \lambda_2 = 1 \\[6pt] p^1_{jj'} = \begin{Vmatrix} 0 & 0 \\ 0 & 6 \end{Vmatrix} \quad p^2_{jj'} = \begin{Vmatrix} 0 & 1 \\ 1 & 4 \end{Vmatrix} \end{array}\right\} \tag{2.08}$$

This is an example of the designs which have been called singular GD designs by Bose, Bose and Shimamoto (Bose, 1951; Bose and Shimamoto, 1952).

From above it is seen that different solutions of (2.04) on invertion turn into different designs. This in a sense justifies the discussion of converse problems.

3. Inversion of Resolvable Designs.

Following Bose (1942) let us define a resolvable incomplete block design (not necessarily balanced or partially balanced) with v, b, r, k as an arrangement of $v.r.$ units of v varieties, there being r units of each variety, into b blocks of size k each, such that the b blocks can be arranged into r sets of n blocks each, each set of blocks containing a complete replication of all varieties. A resolvable incomplete block design will be called affine resolvable when each block of a set has equal number of varieties in common with any other block not in the set. Thus for resolvable designs we must have $v = nk$ and $b = nr$; but for affine resolvability k must also be divisible by n as the k varieties contained in a block are to be equally distributed among the n blocks of any other set. For affine resolvable designs, therefore we must have $v = n^2m$, $b = nr$, r, $k = nm$, where m is an integer equal to k^2/v. Bose (1942) has proved that for affine resolvable balanced incomplete block design we must also have $b = v+r-1$. Plackett and Burman (1946) have proved that affine resolvable incomplete block designs for which $b = v+r-1$ are always balanced. Thus, therefore, for affine resolvable partially balanced incomplete block designs we must have $k^2/v = m$, an integer but $b \neq v+r-1$.

Let us now prove a result which is more general than the result (3) stated in the introduction. In all our discussions it will be tacitly assumed that more than one unit of a variety can not occur in a block.

Theorem 3.1 *Inversion of an affine resolvable incomplete block design with parameters*

$$v = n^2m,\ b = nr,\ r,\ k, = nm \tag{3.01}$$

will always result in a p. b. i. b. design with parameters

$$\left.\begin{array}{l} v' = nr, \quad b' = n^2m, \quad r' = nm, \quad k' = r \\ n_1 = b-n, \ n_2 = n-1, \ \lambda_1 = m, \ \lambda_2 = 0 \\ p^1_{jj'} = \left\Vert \begin{array}{cc} b-2n & n-1 \\ n-1 & 0 \end{array} \right\Vert, \quad p^2_{jj'} = \left\Vert \begin{array}{cc} b-n & 0 \\ 0 & n-2 \end{array} \right\Vert \end{array}\right\} \tag{3.02}$$

Proof. Let B_{ij} be the i-th block belonging to the j-th replication of (3.01). Now B_{ij} has exactly m treatments in common with any other block B_{pq}, $p = 1, 2, \ldots n$, $q = 1, 2, \ldots j-1, j+1, \ldots r$ and no treatment in common with any of the blocks B_{pj}, $p = 1, 2, \ldots i-1, i+1, \ldots n$. Therefore in the inverted design B_{ij} will occur exactly $\lambda_1 = m$ times with each of $n_1 = b-n$ blocks B_{pq}, $p = 1, 2 \ldots n$, $q = 1, 2, \ldots j-1, j+1 \ldots r$ and $\lambda_1 = 0$ time with each of the $n_2 = n-1$ blocks B_{pj}, $p = 1, 2, \ldots i-1, i+1, \ldots n$. Now each of any two blocks belonging to the same replication has exactly m treatments in common with each of the $b-n$ blocks belonging to the other replications. Similarly each of any two blocks belonging to the two different replications has exactly m treatments in common with each of the $b-2n$ blocks of the remaining $r-2$ replications. From these facts the proof of the theorem follows.

Corollary, 3.11. *Any set of L replications $(2 \leq L \leq r)$ of an affine resolvable b. i. b design or p. b. i. b. design on inversion will always result in p. b. i b. design.*

In this connection it is interesting to note that there are affine resolvable incomplete block designs which are neither balanced nor partially balanced. For an example let us consider the following affine resolvable incomplete block design.

$$\left.\begin{array}{ll} B_1\,(1, 3, 4, 5, 9, 11); & B_4\,(2, 6, 7, 8, 10, 12) \\ B_2\,(1, 2\ \ 4, 5, 6, 10); & B_5\,(3, 7, 8, 9, 11, 12) \\ B_3\,(1, 4, 8, 9, 10, 12); & B_6\,(2, 3, 5, 6, 7, 11). \end{array}\right\} \qquad (3.03)$$

Blocks B_i, B_{i+3}, $i = 1, 2, 3$ belong to a replication. It can easily be verified that each block of a replication has exactly three varieties in common with each block not in the replication; but variety 5 does not occur with any of the two varieties 8 and 12 whereas the variety 1 does not occur only with one variety 7. Thus the affine resolvable incomplete block design (3.03) can not be either balanced or partially balanced. But from the theorem proved above the inverted (3.03) design will be a p. b. i. b. design with parameters

$$\left.\begin{array}{l} v = 6, \quad b = 12, \quad r = 6, \quad k = 3 \\ n_1 = 4, \ n_2 = 1, \quad \lambda_1 = 3, \ \lambda_2 = 0 \\ p^1{}_{jj'} = \begin{Vmatrix} 2 & 1 \\ 1 & 0 \end{Vmatrix}, \quad p^2{}_{jj'} = \begin{Vmatrix} 4 & 0 \\ 0 & 0 \end{Vmatrix} \end{array}\right\} \qquad (3.04)$$

Let us now show that there is a class of affine resolvable incomplete arrangements which can not be other than p. b. i. b. designs.

Theorem 3.2. *For an affine resolvable arrangement for which $v = k^2$ the inequality $b \leq v+r-1$ always holds. Further such an affine resolvable arrangement, if existent, must be an affine resolvable p. b. i. b. design unless $b = v+r-1$ in which case the arrangement must be an affine resolvable b. i. b. design.*

Proof. For affine resolvable arrangements $v = n^2m$, $b = nr$, r, $k = nm$. If $v = k^2$, then $m = 1$ and therefore $v = n^2$, $b = nr$, r, $k = n$. Let B_{ij} be the j-th block of the i-th replication, when the blocks are arranged in such a way that the block B_{ij}, $i \neq 1$ contains the variety Q_j, where Q_j is the j-th variety contained in any arbitrarily selected block B_{11}, the k varieties contained in B_{11} being $Q_1, Q_2 \ldots, Q_n$. Of necessity any two of the blocks B_{ij}, $i=1, 2 \ldots r$ must have just one variety Q_j in common between them. Therefore in the whole arrangement Q_j occurs only once with each of the $r(n-1)$ varieties and does not occur with any of the remaining $n^2-r(n-1)-1$ varieties. Here it is to be noted that as $n^2-r(n-1)-1$ can not be negative we must have $r \leq n+1$ and $b \leq v+r-1$. As the block B_{ij} has exactly one variety in common with the block $B_{i'j'}$, $i'=2, 3, \ldots i-1, i+1, \ldots, r$ and none with $B_{ij'}$, for any j', $j' \neq j$, there are exactly $(r-1)(r-2)+n-2$ $(=p^1{}_{11}$, say) varieties with each of which both the varieties Q_j and $Q_{j'}$, where Q_j and $Q_{j'}$ are any two varieties occurring together in any arbitrarily selected block B_{11} occurs seperately. On the other hand if Φ be any variety which does not occur at all with Q_j must occur in the r blocks of r replications, each of which has exactly one variety in common with $r-1$ blocks of different replications in which Q_j occurs. Therefore there are exactly $r(r-1)$ $(= p^2{}_{11}$, say) varieties with which each of Q_j and Φ occurs in the whole design. From these considerations it follows that an affine resolvable incomplete arrangment with $v = k^2$ and $b < v+r-1$ must always be affine resolvable p. b. i. b. design with parameets

$$\left.\begin{aligned} & v = n^3, \quad b = nr, \quad r, \; k = n \\ & n_1 = r(n-1), \; n^2 = (n-1)(n-r+1) \\ & \lambda_1 = 1, \quad \lambda_2 = 0 \\ & p^1_{jj'} = \begin{Vmatrix} (r-1)(r-2)+n-2 & (r-1)(n-r+1) \\ (r-1)(n-r+1) & (n-r)(n-r+1) \end{Vmatrix} \\ & p^2_{jj'} = \begin{Vmatrix} r(r-1) & r(n-r) \\ r(n-r) & (n-r)^2+r-2 \end{Vmatrix} \end{aligned}\right\} \tag{3.05}$$

When $b = v+r-1$, $n_2 = 0$ and $r = n+1$, and therefore the arrangement is an affine resolvable b.i.b. design This last result also follows from the general result obtained by Plackett and Burman (1946), which states that any affine resolvable incomplete arrangement for which $b = v+r-1$ is a balanced one.

It may be noted that the designs (3.05) are same as the ordinary lattice designs.

Corollary 3.21. *Any affine resolvable arrangement with* $v = b = n^2$, $r = k = n$ *is a self-dual affine resolvable p.b.i.b. design with parameters*

$$v = b = n^2, \quad r = k = n, \quad n_1 = n(n-1), \quad n_2 = n-1$$

$$\lambda_1 = 1, \quad \lambda_2 = 0$$

$$p^1_{jj'} = \begin{Vmatrix} n(n-2) & n-1 \\ n-1 & 0 \end{Vmatrix}, \qquad p^2_{jj'} = \begin{Vmatrix} n(n-1) & 0 \\ 0 & n-2 \end{Vmatrix}$$

The proof follows trivially from the theorems (3.1) and (3.2).

Theorem 3.3. *If we have an incomplete block design (not necessarily partially balanced) with* v, b, r, k, *such that the* v *varieties can be divided into* k *groups of* n *varieties each (implying* $v = nk$*), with the property that no two varieties occurring in a group occur together in a block, then on inversion the design will turn into a resolvable design. Further if the original design is affine resolvable, then the inverted design will be a resolvable p.b.i.b. design.*

Proof. We have $v = nk$ and $b = nr$. Let $Q_{i1}, Q_{i2} \dots Q_{in}$ be the n varieties belonging the group G_i, $i = 1, 2, \dots k$ Now with respect to the n varieties $Q_{i1}, Q_{i2}, \dots Q_{in}$ we can always divide the b blocks into n sets of r blocks each, in such a way that in the n blocks of a set only a particular variety Q_{ij}, say, out of the n varieties $Q_{i1}, Q_{i2}, \dots, Q_{in}$ occurs. This shows that in the inverted design the contents of the blocks $Q_{i1}, Q_{i2} \dots, Q_{in}$ will be exclusive as well as exhaustive of all the b varieties. This being true for any i, the inverted design is always a resolvable one. Further if the design is affine resolvable, then, by the theorem 3.1, the inverted design is also a p.b.i.b. design. Therefore the theorem is proved.

Corollary 3.31. *Inversion of a p.b.i.b. design with parameters (3.02) will always result in an affine resolvable incomplete block design with parameters* $v = n^2m$, $b = nr$, r, $k = nm$.

Proof. The design (3.02) is such that we can divide the v' varieties into r groups of n varieties each, such that any two varieties of a group are second associates with $\lambda_2 = 0$ whereas any two varieties belonging to two different groups are first associates. Therefore from the theorem 3.3 it follows that the inverted (3.02) design will be a resolvable design. Now let us show that the inverted (3.02) design can not be resolvable unless it is affine resolvable. As before let us assume $Q_{i1}, Q_{i2}, \ldots, Q_{in}$ to be the n varieties belonging to the i-th group G_i, $i = 1, 2, \ldots k$. Now as Q_{ij} and $Q_{i'j'}$, $i \neq i'$ (for all i i', j, j') occur together exactly in m blocks, and Q_{ij}, $Q_{ij'}$, $j \neq j'$ do not occur together at all, in the inverted design any two blocks Q_{ij} and $Q_{i'j'}$, $i \neq i'$ shall have exactly m varieties in common whereas any two of the blocks $Q_{i1}, Q_{i2}, \ldots Q_{in}$ (constituting the i-th replication) shall have none in common. This completes the proof.

Corollary 3.32. *An affine resolvable p.b.i.b. design with parameters*

$$\left.\begin{array}{c} v = b = n^2 m, \quad r = k = nm \\[1ex] n_1 = n(nm-1), \quad n_2 = n-1, \ \lambda_1 = m, \ \lambda_2 = 0 \\[1ex] p^1_{jj'} = \left\| \begin{array}{cc} b-2n & n-1 \\ n-1 & 0 \end{array} \right\| \quad p^2_{jj'} = \left\| \begin{array}{cc} b-n & 0 \\ 0 & n-2 \end{array} \right\| \end{array}\right\} \qquad (3.06)$$

is a self-dual p. b. i. b. design, i.e., on inversion will remain an affine resolvable p. b. i. b. design with the same parameters.

The proof directly follows from the theorem 3.1 and the corollary 3.31.

In this connection it is interesting to note that an affine resolvable incomplete block design with $v = b = n^2 m$, $r = k = nm$ may not be a p.b.i.b. design as can be seen from the following example which is an affine resolvable arrangement with $v = b = 12$, $r = k = 6$.

1. (1, 3, 4, 5, 9, 11)
2. (2, 3, 5, 6, 7, 11)
3. (4, 6, 7, 9, 10, 11)
4. (1, 5, 7, 8, 10, 11)
5. (1, 2, 6, 8, 9, 11)
6. (2, 3, 4, 8, 10, 11)
7. (2, 6, 7, 8, 10, 12)
8. (1, 4, 8, 9, 10, 12)
9. (1, 2, 3, 5, 8, 12)
10. (2, 3, 4, 6, 9, 12)
11. (3, 4, 5, 7, 10, 12)
12. (1, 5, 6, 7, 9, 12)

In this arrangement the variety 1 occurs twice with each of the varieties 2, 3, 4, 6, 7, 10; thrice with each of the varieties 11, 12 and four times with each of 5, 8, 9 whereas the variety 12 occurs thrice with each of the varieties 1 to 10 and no time with 11. Thus it can not be a p.b.i.b. design. But inverting this we get

1. (1, 4, 5, 8, 9, 12)
2. (2, 5, 6, 7, 9, 10)
3. (1, 2, 6, 9, 10, 11)
4. (1, 3, 6, 8, 10, 11)
5. (1, 2, 4, 9, 11, 12)
6. (2, 3, 5, 7, 10, 12)
7. (2, 3, 4, 7, 11, 12)
8. (4, 5, 6, 7, 8, 9)
9. (1, 3, 5, 8, 10, 12)
10. (3, 4, 6, 7, 8, 11)
11. (1, 2, 3, 4, 5, 6)
12. (7, 8, 9, 10, 11, 12)

(3.07)

which is a solution of the p.b.i.b. design

$$\left.\begin{array}{l} v = b = 12, \quad r = k = 6 \\ n_1 = 10 \quad n_2 = 1, \quad \lambda_1 = 3, \quad \lambda_2 = 0 \\ p^1_{jj'} = \left\|\begin{array}{cc} 8 & 1 \\ 1 & 0 \end{array}\right\|, \quad p^2_{jj'} = \left\|\begin{array}{cc} 10 & 0 \\ 0 & 0 \end{array}\right\| \end{array}\right\} \tag{3.08}$$

due to the theorem 3.1.

The solution (3.0) reveals one interesting property. It has only one set of 2 blocks, *viz*, the blocks no. 11 and 12 containing a complete replication of all varieties and it may be noted that each of the blocks 11 and 12 has exactly three varieties in common with each of the blocks not in the replication. In fact we can prove

Theorem 3.4. *If a p.b.i.b. design with parameters*

$$\left.\begin{array}{l} v = nk, \quad b = n^2m, \quad k, \quad r = nm \\ n_1 = n(k-1), \quad n_2 = n-1, \quad \lambda_1 = m, \quad \lambda_2 = 0 \\ p^1_{jj'} = \left\|\begin{array}{cc} v-2n & n-1 \\ n-1 & 0 \end{array}\right\|, \quad p^2_{jj'} = \left\|\begin{array}{cc} v-n & 0 \\ 0 & n-2 \end{array}\right\| \end{array}\right\} \tag{3.09}$$

has a solution with a set of n blocks containing a complete replication of all varieties, such that any block belonging to the replication has equal number of varieties in common with each of the blocks not in the set, then we must have k = nm and the number of varieties in common will be m.

Conversely, if for a p.b.i.b. design (3.09) having a set of n blocks containing a complete replication of all varieties k = nm, then any block of the set must have equal number of varieties (namely m) in common with each of the blocks not in the set.

Proof. Let $B_{01}, B_{02}, \ldots, B_{0n}$ be a set of n blocks with a complete replication of all varieties and $B_1, B_2, \ldots B_{b-n}$ be the remaining $b-n = n(r-1)$ blocks. If a_j be the number of varieties common between any two blocks B_{0i} (for any fixed i, $1 \leq i \leq n$) and B_j $(1 \leq j \leq b-n)$, then

$$\sum a_j = k(r-1) \tag{3.10}$$

as each of the k varieties occurring in B_{0i} will occur $r-1$ times in the remaining $b-n$ blocks; and

$$\sum a_j(a_j-1) = k(k-1)(m-1), \tag{3.11}$$

as any two of the k varieties of a block can not be other than first associates and so will occur together $\lambda_1 - 1 = m-1$ times in the $b-n$ blocks not in the set.

From (3.10) and (3.11) we have

$$(b-n)V(a_j) = \sum a_j^2 - \frac{(\sum a_j)^2}{b-n} = \frac{(n-1)(nm-k)k}{n} \tag{3.12}$$

(3.12) shows that $V(a_j) = 0$ when and only when $k = nm$. This proves our theorem.

Here it may be noted that as $V(a_j)$ can not be negative there can not be a p.b.i.b. design (3.09) with a set of n blocks containing a complete replication of all varieties if $r < k$.

It is interesting to compare the theorem 3.4 with the following result:

If a b.i.b. design belonging to the series (P) $v = nk$, $b = n(n\lambda+g)$, $r = n\lambda+g$, k, λ *where* g *is an integer given by* $(n-1)\lambda/(k-1)$ *has a solution with a set of blocks providing a complete replication of all the varieties such that any block of the replication has equal number of varities in common with each of the blocks not in the set, then we must have* $b = v+r-1$ *and* λ *of the form* $nt+1$.

Conversely if for a b.i.b. design (P) having a set of n blocks containing a complete replication of all varieties $b = v+r-1$, *then* λ *should be of the form* $nt+1$ *and any block of the set must have equal number of varieties (namely* k^2/v*) in common with each of the blocks not in the set (Roy, 1952).*

4. Inversion of Some P. B. I. B. Designs.

Partially balanced incomplete block designs which are inverted balanced incomplete block designs on re-inversion will obviously turn into b.i.b. designs. But will the p.b.i.b. designs corresponding to the results (1), (2), (3) or (4) but not obtained by inversion on inversion always turn into b.i.b. designs? Here it will be shown that in case of the results (1) and (3), the converse is also true.

Theorem 4.1. *Inversion of a p.b i.b. design with parameters*

$$\left.\begin{array}{c} v' = b, \quad b' = v, \quad r' = k, \quad k' = r \\ n_1 = k(r-1), \quad n_2 = b-k(r-1)-1 \\ \lambda_1 = 1, \quad \lambda_2 = 0 \\ p^1_{ij'} = \left\Vert\begin{matrix} r-2+(k-1)^2 & (r-k)(k-1) \\ (r-k)(k-1) & b+r-k(2r-k)-1 \end{matrix}\right\Vert, \\ p^2_{ij'} = \left\Vert\begin{matrix} k^2 & k(r-k-1) \\ k(r-k-1) & b+k^2-2\{k(r-1)+1\} \end{matrix}\right\Vert \end{array}\right\} \tag{4 01}$$

will always result in a b. i. b. design with parameters v, b, r, k, $\lambda = 1$. (4.02)

Proof. As $\lambda_1 = 1$, no two varieties can occur together more than once. Therefore every block of the p. b. i. b. design with (4.01) has one variety in common with each of $k'(r'-1) = r(k-1)$ blocks of the design Now from the relation $n_1 p^1_{21} = n_2 p^2_{11}$, on simplification we get $v-1 = r(k-1)$ i.e. $b'-1=r(k-1)$. Therefore the p. b. i. b. design with parameters (4.01) will always be a singly linked block design, that is, any two blocks of the design must have exactly one variety in common. This shows that on inversion the p. b, i. b. design (4.01) will always turn into the b. i. b. design (4.02).

Corollary, 4.11. *The total number of independent solutions for a p. b. i. b. design with parameters (4.01) are equal to the total number of independent solutions of a b. i. b. design with parameters (4.02).*

Fisher (1940) has worked out 79 sets of independent solution for the b. i. b. design $v = 15,\ b = 35,\ r = 7,\ k = 3,\ \lambda = 1$. Therefore the p. b. i. b. design

$$\left.\begin{array}{l} v = 35, \quad b = 15, \quad r = 3 \quad k = 7 \\[2ex] n_1 = 18,\ n = 16,\ \lambda_1 = 1,\ \lambda_2 = 0 \\[2ex] p^1_{jj'} = \begin{Vmatrix} 9 & 8 \\ 8 & 8 \end{Vmatrix} \ p^2_{jj'} = \begin{Vmatrix} 9 & 9 \\ 9 & 6 \end{Vmatrix} \end{array}\right\}$$

has also 79 sets of independent solutions which can be obtained by inverting the solutions of the b. i. b. design $v = 15,\ b = 35,\ r = 7,\ k = 3,\ \lambda = 1$.

Theorem 4.2. *Inversion of a p. b. i. b. design with parameters*

$$\left.\begin{array}{l} v = n(n^2t+n+1), \quad b = n^2(n-1)t+n^2, \quad r = n(n-1)t+n, \\[2ex] k = n^2t+n+1,\ n_1 = n^2(nt+1),\ n_2 = n-1,\ \lambda_1 = (n+1\ t,\ \lambda_2 = 0 \\[2ex] p^1_{jj'} = \begin{Vmatrix} n(n^2t+n-1) & n-1 \\ n-1 & 0 \end{Vmatrix} \ p^2_{jj'} = \begin{Vmatrix} n^2(nt+1) & 0 \\ 0 & n-2 \end{Vmatrix} \end{array}\right\} \tag{4.03}$$

will always lead to an affine resolvable b. i. b. design with parameters

$$\left.\begin{array}{l} v' = n^2(n-1)t+n^2, \quad b' = n(n^2t+n+1), \quad r' = n^2t+n+1 \\[2ex] k' = n(n-1)t+n, \quad \lambda = nt+1. \end{array}\right\} \tag{4.04}$$

Proof. Let us choose any initial block B_0 and let the other blocks be $B_1, B_2 \ldots, B_{b-1}$. We shall calculate the variance σ^2 of the set of integers $a_1, a_2, \ldots a_{b-1}$, where a_i is the number of varieties common between B_0 and B_i. As each of the k varieties occurring in B_0 will occur '$r-1$' times in the remaining $b-1$ blocks, we have

$$\sum a_i = k(r-1) \tag{4.05}$$

Since any two of the k varieties of any block can not be other than first associates, any pair of the varieties in the initial block will occur exactly $\lambda_1 - 1$ times in the $b-1$ blocks and therefore we have

$$\sum a_i(a_i-1) = k(k-1)(\lambda_1-1) \tag{4.06}$$

From (4.05) and (4.06) we get

$$(b-1)V(a_i) = \sum a^2_i - \frac{(\sum a_i)^2}{b-1}$$

$$= k(k-1)(\lambda_1-1) + k(r-1) - \frac{k^2(r-1)^2}{b-1} \tag{4.07}$$

Uptil now we have not utilised any property of p. b. i. b design except that the p. b. i. b. design is a design with two associate classes, for which λ_1 is an integer and $\lambda_2 = 0$. Thus we can state that if the parameters of a p.b.i.b. design with $v,\ b,\ r,\ k,\ \lambda_1,\ \lambda_2, = 0$ are such that the R. H. S. of (4.07) is zero, then the p. b. i. b, design is a times linked one, where $a = a_1 = a_2 = \ldots\ldots = a_{b-1} = \dfrac{k(r-1)}{b-1}$, an integer, and therefore on inversion the p. b. i. b. design will turn into a b. i. b. design with $\lambda = a$. This fact can also be utilised to prove our theorem 4.1. In the present case it can be verified that $(b-1)V(a)$ is zero and

$\frac{k(r-1)}{b-1} = nt+1$. Thus on inversion (4.03) will always turn into the b. i. b. design (4.04). But from the corollary 3.31 we know that inverted (4.03) design will always be affine resolvable. This completes the proof of our theorem

Corollary 4.21. *All the b. i. b. designs belonging to the series* (O_1) $v = s^2$, $b = s^2+s$, $r = s+1$, $k = s$, $\lambda = 1$, *if existent, must be affine resolvable b. i. b. designs.*

This follows from the theorems 4.1 and 4.2.

Corollary 4.22 *The totality of solutions for the p. b. i. b. design (4.03) is identical with the totality of affine resolvable solutions for the b. i. b. design (4.04).*

Thus the p. b. i. b. design

$$v = 14,\ b = 8,\ r = 4,\ k = 7,\ n_1 = 12,\ n_2 = 1.$$

$$\lambda_1 = 2,\ \lambda_2 = 0.$$

$$p^1_{jj'} = \begin{Vmatrix} 10 & 1 \\ 1 & 0 \end{Vmatrix} \quad p^2_{jj'} = \begin{Vmatrix} 12 & 0 \\ 0 & 0 \end{Vmatrix}$$

has only one solution as it has been shown by Nandi (1946) that the b. i. b design

$v = 8$, $b = 14$, $r = 7$, $k = 4$, $\lambda = 3$ has only one affine resolvable solution (namely, the solution $[\gamma]$ of page 314).

Corollary 4.23. *Affine resolvable incomplete designs for which* $b = v+r-1$ *are always balanced.*

This follows from the theorems 3.1 and 4.2.

It is interesting to note that the result of corollary 4.23 has also been obtained by Plackett and Burman (1946). In fact we can take the theorem (4.2) as a particular case of the corollary 3.31 if we assume the result due to Plackett and Burman for granted.

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ON SOME PROPERTIES OF THE BESSEL FUNCTION DISTRIBUTIONS

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Introduction. At present the Bessel function distributions as introduced by McKay (1932) play an important role in the realm of graduation particularly for that region of the Pearsonian $\beta_1 - \beta_2$ plane which lies below the Type III line.

It is also, wellknown that the distribution of the classical D^2-statistic based on p characters as obtained by R. C. Bose (1936) involves the Bessel function of the form

$$\text{const } (D_p^2)^{\frac{1}{4}(p-2)} e^{-\frac{1}{2}cD_p^2} . \quad I_{\frac{1}{2}(p-2)}\left[\frac{c}{2}\sqrt{(D_p^2\Delta_p^2)}\right] d(D_p^2)$$

where $c = n_1n_2/(n_1+n_2)$, where n_1 and n_2 represent the sample sizes for the two populations and Δ_p^2 being the population value of D_p^2. Next S. S. Bose (1937) made a critical study of the Bessel function distributions (particularly I-type) and discussed its suitability for the purposes of graduation.

The aim of the present paper is to solve some distribution problems connected with the Bessel variates by the application of the method of characteristic functions. Some of the results derived in the paper are not new and obtained earlier by some other analytical or geometrical methods by various authors. But still this paper serves as an interesting illustration as to how the technique of the characteristic functions can be employed as a powerful tool in solving the distribution problems in statistics.

Section I. *The distribution of the sum of a number of independent Bessel variates*:—We shall take the form of the I-type Bessel function distribution as

$$dF(x) = \text{const. } x^{\frac{1}{2}(p-1)} e^{-\alpha x} I_{p-1}(\beta x^{\frac{1}{2}}) dx \quad (0 \leq x < \infty)$$

where $I_{p-1}(\beta x^{\frac{1}{2}})$ is the modified Bessel function of the first kind as defined by Watson (1922),

Now the c. f for the distribution of the Bessel variate x is given by

$$\varphi_x(t) = \int_0^\infty e^{itx} dF(x). \tag{1.1}$$

But since

$$I_{p-1}(\beta x^{\frac{1}{2}}) = \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(p+r)} \cdot \left(\frac{\beta x^{\frac{1}{2}}}{2}\right)^{p-1+2r} \tag{1.2}$$

substituting (1.2) in (1.1) we have

$$\varphi_x(t) = \text{const.} \left(\frac{\beta}{2}\right)^{p-1} \cdot \frac{1}{(\alpha - it)^p} \cdot \exp\left[\frac{\beta^2}{4(\alpha - it)}\right] \tag{1.3}$$

Now putting $t = 0$ since

$$\varphi(0) = \int_0^\infty dF(x) = 1$$

$$(2/\beta)^{p-1}\alpha^p . e^{-\frac{1}{4}\beta^2/\alpha} = \text{const.} \tag{1.4}$$

Hence the distribution of the Bessel variate x will be given by

$$dF(x) = (2/\beta)^{p-1} x^{\frac{1}{2}(p-1)} \alpha^p \cdot e^{-\frac{1}{4}\beta^2/\alpha} \cdot e^{-\alpha x} I_{p-1}(\beta x^{\frac{1}{2}}) dx \tag{1.5}$$

and the corresponding characteristic function becomes

$$\varphi_x(t) = \left(1 - \frac{it}{\alpha}\right)^{-p} \cdot \exp \frac{it\beta^2}{4\alpha^2(1 - it/\alpha)}. \tag{1.6}$$

Let us now suppose that there are n variates $x_j (j = 1, 2, \ldots, n)$ with the distribution function for x_j as

$$dF(x_j) = \text{const. } x_j^{\frac{1}{2}(p_j - 1)} e^{-x_j} I_{p_j - 1}(\beta_j x_j^{\frac{1}{2}}) dx_j \quad j = 1, 2, \ldots n. \tag{1.7}$$

then we have

$$\varphi_{x_j}(t) = (1 - it)^{-p_j} \exp\left[\frac{it\beta_j^2}{4(1 - it)}\right] \tag{1.8}$$

Now it is wellknown that the c. f. for the distribution of the sum of several independent random variables is equal to the product of their separate characteristic functions and hence the c. f. for the distribution of the sum $u = x_1 + x_2 + \cdots + x_n$ is given by

$$\varphi_u(t) = \prod_{j=1}^{n} \varphi_{x_j}(t) = (1 - it)^{-p_0} \exp\left[\frac{it\beta_0^2}{4(1 - it)}\right] \tag{1.9}$$

where

$$p_0 = \sum_{j=1}^{n} p_j \quad \text{and} \quad \beta_0^2 = \sum_{j=1}^{n} \beta_j^2.$$

Now by applying the Fourier Integral theorem, the frequency function of u is given by

$$f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iut} \varphi_u(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - it)^{-p_0} dt \exp\left\{\frac{it\beta_0^2}{4(1 - it)} - iut\right\}. \tag{1.10}$$

In order to evaluate (1 10) let us first of all find the value of the integral

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 - \frac{it}{b}\right)^{-d} \cdot \exp\left[\frac{ita}{(1 - it/b)} - itc\right]$$

where

$$(a \geq 0; \quad b, c, d > 0). \tag{1.11}$$

Now putting $1 - it/b = -z/c$ the integral (1.11) reduces to the form

$$I = \frac{1}{2\pi i} . b . c^{d-1} . e^{-b(a+c)} \int_{-c-i\infty}^{-c+i\infty} \exp\left\{-bz - \frac{abc}{z}\right\} . (-z)^{-d} dz$$

$$= \frac{i}{2\pi} b^d . c^{d-1} e^{-b(a+c)} . \int_{-bc+i\infty}^{-bc-i\infty} \exp\left(-w - \frac{ab^2c}{w}\right) . (-w)^{-d} dw$$

(where $w = bz$)

$$= \frac{i}{2\pi} b^d c^{d-1} e^{-b(a+c)} \sum_{r=0}^{\infty} \frac{(ab^2c)^r}{r!} \int_{-bc+i\infty}^{-bc-i\infty} e^{-w} (-w)^{-(d+r)} dw$$

But the integral in the last expression is

$$\frac{i}{2\pi}\int_{-bc+i\infty}^{-bc-i\infty} e^{-w}(-w)^{-(d+r)}\,dw = \frac{1}{\Gamma(d+r)} \qquad (bc>0)$$

the above integral being the Hankel's expression for $\Gamma(z)$ as a contour integral *cf.* Whittaker and Watson (1920, p. 245). Hence the required integral is

$$I = b^d c^{d-1} e^{-b(a+c)} \sum_{r=0}^{\infty} \frac{\{b(ac)^{\frac{1}{2}}\}^{2r}}{r\,!\,\Gamma(d+r)} \tag{1. 13}$$

Hence from the definition of a Bessel function, it obviously follows that

$$I = b\left(\frac{c}{a}\right)^{\frac{1}{2}(d-1)} e^{-b(a+c)} I_{d-1}\{2b(ac)^{\frac{1}{2}}\} \tag{1. 14}$$

Now substituting $a = \frac{1}{4}\beta_0^2$; $b = 1$; $c = u$ and $d = p_0$ in (1 10) we have at once

$$f(u) = \left(\frac{4u}{\beta_0^2}\right)^{\frac{1}{2}(p_0-1)} \exp\{-u-\tfrac{1}{4}\beta_0^2\}\cdot I_{p_0-1}(\beta_0 u^{\frac{1}{2}}). \tag{1. 15}$$

Thus we have established the reproductive property of the Bessel variates, that is the distribution of the sum of a number of I-type independent Bessel variates is also a Bessel variate.

Corollary 1. The distribution of the sample mean from a Bessel population.

Let x_1, x_2......x_n represent a sample of size n from a I-type Bessel function distribution given by

$dF(x) = \text{const. } x^{p-1}e^{-\alpha x} I_{p-1}(\beta x^{\frac{1}{2}})$ with its

$$\text{c. f.} \quad \varphi_x(t) = \left(1-\frac{it}{\alpha}\right)^{-p} \exp\left[\frac{it\beta^2}{4\alpha^2(1-it/\alpha)}\right]$$

If $\bar{x} = (x_1+x_2+\ldots\ldots+x_n)/n$ represents the sample mean, then the c. f. for $\bar{x}$ is given by

$$\varphi_{\bar{x}}(t) = \{\varphi_x(t/n)\}^n = \left(1-\frac{it}{n\alpha}\right)^{-pn} \times \exp\frac{it\beta^2}{4\alpha^2(1-it/n\alpha)} \tag{1. 16}$$

Hence

$$f(\bar{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{\frac{it\beta^2}{4\alpha^2(1-it/n\alpha)} - it\bar{x}\right\}\left(1-\frac{it}{n\alpha}\right)^{-np} dt \tag{1. 17}$$

Now putting $a = \frac{1}{4}\beta^2/\alpha^2$; $b = n\alpha$; $c = \bar{x}$ and $d = np$ in the integral (1. 10), the frequency function for the sample mean $\bar{x}$ becomes

$$f(\bar{x}) = n\alpha\left(\frac{4\alpha^2\bar{x}}{\beta^2}\right)^{\frac{1}{2}(np-1)} \exp\left\{-n\alpha\left(\frac{\beta^2}{4\alpha^2}+\bar{x}\right)\right\} I_{np-1}(n\beta\sqrt{\bar{x}})$$

$$= \text{const. } \bar{x}^{\frac{1}{2}np-1} e^{-n\alpha\bar{x}} I_{np-1}(n\beta\sqrt{\bar{x}}) \tag{1. 18}$$

The distribution (1. 18) has been already derived by R. C. Bose (1936) by analytical method.

Corollary 2. The distribution of the mean from a Gamma population

Let $x_1\ x_2 \ldots x_n$ be a sample of size n drawn independently at random from a Gamma Population $dF(x) = \text{const}\ x^{p-1}e^{-\alpha x}dx$.

Now we note obviously that as $\beta \to 0$ the distribution function tends to the Gamma distribution. Hence the distribution of the sample mean $\bar{x}$ will be obviously given by $f(\bar{x}) = \text{const.}\ \bar{x}^{np-1}e^{-n\alpha\bar{x}}$ on taking the limiting form of $f(\bar{x})$ as $\beta \to 0$ in the expression (1. 18).

This has been already derived by J. O. Irwin (1927).

Corollary 3. The distribution of the non-central χ^2,

Let $x_1, x_2 \ldots x_n$ be n independent normal variates where x_j is distributed normally with the mean m_j and variance σ^2 ($j = 1, 2, \ldots n$). In such a case the non-central χ^2 is is usually defined by the sum $\chi'^2 = \frac{1}{\sigma^2}\sum_{j=1}^{n} x_j^2$

If now x is a normal variate $N(m, \sigma)$ then the c. f. for the distribution of x^2 is given by

$$\varphi_{x^2}(t) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left\{it\,x^2 - \tfrac{1}{2}\ \frac{(x-m)^2}{\sigma^2}\right\}dx$$

$$= (1-2\,it\,\sigma^2)^{-\frac{1}{2}}\ \exp\left\{\frac{it\,m^2}{1-2\,it\,\sigma^2}\right\} \tag{1. 19}$$

Hence when x_j is a $N(m_j;\,\sigma)$ variate for $j = 1, 2 \ldots n$ and $\chi'^2 = \frac{1}{\sigma^2}\sum_{j=1}^{n} x_j^2$ the characteristic function for the distribution of χ'^2 will be given by

$$\varphi_{\chi'^2}(t) = (1-2\,it)^{-\frac{1}{2}n}\ \exp\left\{\frac{it\,m^2/\sigma^2}{1-2\,it}\right\} \text{ where } m^2 = \sum_{j=1}^{n} m_j^2 \tag{1. 20}$$

Then by Fourier Integral theorem the distribution of the non-central χ^2 is given by

$$f(\chi'^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{\frac{it\,m^2/\sigma^2}{1-2\,it} - it\,\chi'^2\right\}(1-2\,it)^{-\frac{1}{2}n}\,dt \tag{1. 21}$$

Now putting $a = \frac{m^2}{\sigma^2}$; $b = \frac{1}{2}$; $c = \chi'^2$ and $d = \frac{1}{2}n$ in the integral (1. 11) the frequency function for the non-central χ^2 as given by the solution of the definite integral in (1. 14) becomes

$$f(\chi'^2) = \left(\frac{\chi'^2}{\lambda^2}\right)^{\frac{1}{2}(n-2)} \exp\{-\tfrac{1}{2}(\chi'^2+\lambda^2)\}\ I_{\frac{1}{2}(n-2)}(\chi'^2\,\lambda^2)^{\frac{1}{2}} \tag{1. 22}$$

where

$$\lambda^2 = \frac{1}{\sigma^2}\sum_{j=1}^{n} m_j^2.$$

Thus we see that the noncentral χ^2 is a Bessel variate and hence from the results of Section I it follows evidently that if there are p noncentral χ^2's with parameters λ_j^2 and n_j d.f.'s ($j = 1, 2 \ldots p$) then the sum $\chi^2 = \chi_1^2 + \cdots + \chi_p^2$ must also follow a noncentral χ^2 distribution with the parameter $\lambda^2 = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_p^2$ and $n = n_1 + n_2 + \cdots + n_p$ d. f. This reqroductive property of the noncentral χ^2 distribution has been shown by P. C. Tang (1938) by analytical methods.

Section II. *The distribution of the ratio of two independent Bessel Variates.*

Let us consider two independent Bessel variates x_1 and x_2 with their distribution functions respectively given by

$$dF(x_j) = \text{const. } x_j^{\frac{1}{2}(p_j-1)} e^{-\alpha_j x_j} I_{p_j-1}(\beta_j x_j^{\frac{1}{2}}) dx_j, \quad \text{for } j = 1, 2. \tag{2.1}$$

We now proceed to find the distribution of $u = x_1/x_2$. We shall first of all derive the distribution of $\log u = \log x_1 - \log x_2 = w$ (say)

Then the characteristic function for the distribution of w will be given by

$$\varphi_w(t) = \int_0^\infty \int_0^\infty e^{iwt}\, dF(x_1).dF(x_2) = K \int_0^\infty x_1^{\frac{1}{2}(p_1-1)+it} e^{-\alpha_1 x_1} I_{p_1-1}(\beta_1 x_1^{\frac{1}{2}}) dx_1$$

$$\times \int_0^\infty x_2^{\frac{1}{2}(p_2-1)-it} e^{-\alpha_2 x_2} I_{p_2-1}(\beta_2 x_2^{\frac{1}{2}}) dx_2 \tag{2.2}$$

Now evaluating the integrals and substituting the value of K

$$= \left(\frac{2}{\beta_1}\right)^{p_1-1} \cdot \left(\frac{2}{\beta_2}\right)^{p_2-1} \alpha_1^{p_1} . \alpha_2^{p_2} \exp\left\{-\frac{1}{4}\left(\frac{\beta_1^2}{\alpha_1} + \frac{\beta_2^2}{\alpha_2}\right)\right. \tag{2.2}$$

in (2.2) we obtain

$$\varphi_w(t) = \exp\left\{-\frac{1}{4}\left(\frac{\beta_1^2}{\alpha_1} + \frac{\beta_2^2}{\alpha_2}\right)\right\}$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \left(\frac{\beta_1}{2}\right)^{2r} \left(\frac{\beta_2}{2}\right)^{2s} \cdot \frac{1}{\alpha_1^{r+it}} \cdot \frac{1}{\alpha_2^{s-it}} \times \frac{\Gamma(p_1+r+it)}{r!\Gamma(p_1+r)} \cdot \frac{\Gamma(p_2+s-it)}{s!\Gamma(p_2+s)}. \tag{2.3}$$

In order to avoid unnecessary complications, let us consider the case where $\alpha_1 = \alpha_2 = \alpha$ and there by applying the Fourier Integral theorem, the frequency function for w will be

$$f(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwt} . \varphi_w(t) dt$$

$$= \frac{1}{2\pi} \exp\left\{-\frac{1}{4\alpha}(\beta_1^2 + \beta_2^2)\right\} . \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \left(\frac{\beta_1}{2}\right)^{2r} . \left(\frac{\beta_2}{2}\right)^{2s} . \frac{1}{\alpha^{r+s}}$$

$$\times \frac{1}{r!\Gamma(p_1+r)} . \frac{1}{s!\Gamma(p_2+s)} \int_{-\infty}^{\infty} e^{-iwt} \Gamma(p_1+r+it) . \Gamma(p_2+s-it) dt. \tag{2.4}$$

In order to evaluate the integral in (2.4), let us now substitute $p_2+s-it=-z$ and there we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwt}\Gamma(p_1+r+it).\Gamma(p_2+s-it)dt$$

$$= \frac{1}{2\pi i}\, e^{-w(p_2+s)} \int_{-p_2-s-i\infty}^{-p_2-s+i\infty} e^{-wz}.\Gamma(p_1+p_2+r+s+z)\,\Gamma(-z)dz$$

$$= e^{-w(p_2+s)}.(1+e^{-w})^{-(p_1+p_2+r+s)}.\Gamma(p_1+p_2+r+s). \tag{2.5}$$

[*Cf.* Whittaker and Watson (1920)]

Now substituting this value in (2.4) the frequency function for w becomes

$$f(w) = \exp\left\{-\frac{1}{4\alpha}(\beta_1^2+\beta_2^2)\right\}.\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left(\frac{\beta_1}{2}\right)^{2r}.\left(\frac{\beta_2}{2}\right)^{2s}.\frac{1}{\alpha^{r}+s}$$

$$\times\frac{\Gamma(p_1+p_2+r+s)}{r\,!\,s\,!\,\Gamma(p_1+r)\Gamma(p_2+s)}.\frac{e^{-w(p_2+s)}}{(1+e^{-w})^{p_1+p_2+r+s}} \tag{2.6}$$

But since $u=e^{w}$ we have $dw=du/u$ and hence the frequency function of u becomes

$$f(u) = \exp\left\{-\frac{1}{4\alpha}(\beta_1^2+\beta_2^2)\right\}.\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left(\frac{\beta_1}{2}\right)^{2r}.\left(\frac{\beta_2}{2}\right)^{2s}.\frac{1}{\alpha^{r+s}}$$

$$\times\frac{1}{r\,!\,s\,!}.\frac{1}{B(p_1+r,\ p_2+s)}.\frac{u^{p_1+r-1}}{(1+u)^{p_1+p_2+r+s}}\ . \tag{2.7}$$

Corrollary I: *The distribution of the ratio of two non-central* χ^2*'s.*

It is already shown in the Cor. 8 of the Section I of this paper that the non-central χ^2 is a Bessel variate. Hence if these two non-central χ^2's independently distributed of each other, the ratio $u=\chi_1^2/\chi_2^2$ will follow the distribution (2.7) above.

Thus if χ_i^2 is distributed with parameter λ_i^2 and with n_i d.f. $(i=1,2)$ the distribution of $u=\chi_1^3/\chi_2^2$ after some simplifications will reduce to the form

$$f(u) = \exp\{-\tfrac{1}{2}(\lambda_1^2+\lambda_2^2)\}\times\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left(\frac{\lambda_1^2}{2}\right)^{r}.\left(\frac{\lambda_2^2}{2}\right)^{s}$$

$$\times\frac{1}{r\,!\,s\,!\,B(\frac{1}{2}n_1+r,\ \frac{1}{2}n_2+s)}\times\frac{u^{\frac{1}{2}n_1+r-1}}{(1+u)^{\frac{1}{2}(n_1+n_2)+r+s}} \tag{2·8}$$

Now putting $\beta_1=\beta$ in the distribution (2.1) and taking the limiting form as $\beta_2\to 0$, the frequency function for $u=x_1/x_2$ as given in (2.7) reduces to the form

$$f(u) = \exp\left\{-\frac{\beta^2}{4\alpha}\right\}\sum_{r=0}^{\infty}\left(\frac{\beta_1^2}{4\alpha}\right)^{r}.\frac{1}{r\,!\,B(p_1+r,{}_1p_2)}.\frac{u^{p_1+r-1}}{(1+u)^{p_1+p_2+r}} \tag{2.9}$$

the frequency function for $w=u/1+u$ will be given by

$$f(w) = \exp\left\{-\frac{\beta^2}{4\alpha}\right\}.\frac{1}{B(p_1,\ p_2)}.w^{p_1-1}(1-w)^{p_2-1}.\ {}_1F_1(p_1+p_3\,;\ p_1\,;\ \tfrac{1}{4}\beta^2w/\alpha) \tag{2.10}$$

where

$$ {}_1F_1(\alpha, \beta, x) = 1+\frac{\alpha}{\beta}x+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}.\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}.\frac{x^3}{3!}+\ldots $$

represents the confluent hypergeometric function.

Now from the result in (2.10), the distribution of the ratio of a non-central χ^2 and a central χ^2 follows easily. Some of the results derived in this section has been discussed by P. C. Tang (1938).

Section III. *The distribution of the difference of two independent Bessel Variates.*

Let us now consider the distribution of the difference $u = x_1-x_2$ where x_1 and x_2 are two independent Bessel variates with their respective distribution functions

$$ dF(x_j) = \text{const. } x_j^{\frac{1}{2}(p_j-1)}.\ e^{-x_j}I_{p_j-1}(\beta_j x_j^{\frac{1}{2}})dx_j,\ \ (j = 1, 2). \tag{3.1} $$

Now the characteristic function for the distribution of the variate x_1 will be given by

$$ \varphi_{x_1}(t) = (1-it)^{-p_1}.\ \exp\left[-\tfrac{1}{4}\beta_1^2+\frac{\beta_1^2}{4(1-it)}\right] $$

and similarily for

$$ \varphi_{x_2}(t) = (1-it)^{-p_2}.\ \exp\left[-\tfrac{1}{4}\beta_2^2+\frac{\beta_2^2}{4(1-it)}\right]. \tag{3.2} $$

Then the c. f. for the distribution of the difference $u = x_1-x_2$ is given by

$$ \varphi_u(t) = \exp\{-\tfrac{1}{4}(\beta_1^2+\beta_2^2)\} \times \exp\left\{\frac{1}{4}\left(\frac{\beta_1^2}{1-it}+\frac{\beta_2^2}{1+it}\right)\right\}.\ (1-it)^{-p_1}(1+it)^{-p_2} \tag{3.3} $$

In order to avoid unnecessary mathematical complications, let us now consider the case when $\beta_1 = \beta_2 = \beta$. In such a case the expression (3.3) reduces to the form

$$ \varphi_u(t) = e^{-\frac{1}{2}\beta^2}.\ \exp\left\{\frac{1}{2}.\frac{\beta^2}{(1+it)(1-it)}\right\}.(1-it)^{-p_1}(1+it)^{-p_2} \tag{3.4} $$

Hence the distribution function for u becomes

$$ f(u) = \frac{e^{-\frac{1}{2}\beta^2}}{2\pi}\int_{-\infty}^{\infty}\exp\left\{-iut+\frac{1}{2}.\frac{\beta^2}{(1+it)(1-it)}\right\}.(1-it)^{-p_1}(1+it)^{-p_2} \tag{3.5} $$

Let us now make the substitution $1-it = -z/u$ and then (3.5) gives us

$$ \begin{aligned} f(u) &= \frac{e^{-\frac{1}{2}\beta^2}}{2\pi}\int_{-\infty}^{\infty}\exp\left\{\frac{1}{2}.\frac{\beta^2}{(1+it)(1-it)}-iut\right\}.(1-it)^{-p_1}(1+it)^{-p_2}\,dt \\ &= \frac{e^{-\frac{1}{2}\beta^2}}{2\pi i}.\frac{e^{-u}u^{p_1-1}}{2^{p_2}}\int_{-u-i\infty}^{-u+i\infty}\exp\left\{\frac{\beta^2}{4(-z/u)(1+z/2u)}-z\right\}(-z)^{-p_1}(1+z/2u)^{-p_2}dz \end{aligned} \tag{3.6} $$

Now expanding the exponent as a power series in u, (3.6) gives us

$$ f(u) = \frac{e^{-\frac{1}{2}\beta^2}}{2\pi i}\times\frac{e^{-u}u^{p_1-1}}{2^{p_2}}\sum_{r=0}^{\infty}\frac{1}{r!}.\frac{\beta^{2r}.u^r}{2^{2r}}\int_{-u-i\infty}^{-u+i\infty}\frac{e^{-z}dz}{(-z)^{p_1+r}(1+z/2u)^{p_2+r}}. \tag{3.7} $$

But it is known that

$$\frac{1}{2\pi i}\int_{-u-i\infty}^{-u+i\infty}\frac{e^{-z}dz}{(-z)^{p_1+r}(1+z/2u)^{p_2+r}}=\frac{e^{u}(2u)^{\frac{1}{2}(p_2-p_1)}}{\Gamma(p_1+r)}\cdot W_{\frac{1}{2}(p_1-p_2),\,\frac{1}{2}(1-p_1-p_2-2r)}(2u) \tag{3.8}$$

where $W_{k,m}(u)$ is Whittaker's Function [*Cf.* (Whittaker and Watson (1920), Page 340]. Further we have $W_{k,m}(u) = W_{k,\,-m}(u)$ hence the frequency function of $u = x_1-x_2$ will be given by

$$f(u)=e^{-\frac{1}{2}\beta^2}.\frac{u^{\frac{1}{2}(p_1+p_2)-1}}{2^{\frac{1}{2}(p_1+p_2)}}\sum_{r=0}^{\infty}\frac{\beta^{2r}u^{r}}{r\,!\Gamma(p_1+r).2^{2r}}\cdot W_{\frac{1}{2}(p_1-p_2),\,\frac{1}{2}(p_1+p_2+2r-1)}(2u). \tag{3.9}$$

For the case when $p_1 = p_2 = p$ we have the important relation $W_{0,m}(2u) = (2u/\pi)^{\frac{1}{2}}K_m(u)$ where $K_m(u)$ is modified Bessel function of the second kind as defined by Watson (*c.f.* Whittaker and Watson, 1920, pp. 378) and then (3.9) reduces the form

$$f(u)=e^{-\frac{1}{2}\beta^2}.\frac{u^{\frac{1}{2}(2p-1)}}{2^{\frac{1}{2}(2p-1)}}\cdot\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{2^{2r}u^{r}}{r\,!\Gamma(p_1+r).2^{2r}}\cdot K_{p+r-\frac{1}{2}}(u). \tag{3.10}$$

Corollary 1: The distribution of the product xy.

Let x and y be two independent normal variates with the distribution functions $N(m\,;\sigma)$ and $N(0\,;\sigma')$ respectively. We have to find the distribution of the product xy. Now making the variate transformation $v = x/\sigma+y/\sigma'$ and $w = x/\sigma-y/\sigma'$ it is seen evidently that $xy = \text{const.}\,(v^2-w^2)$ where the distribution functions of v and w are $N(m/\sigma\,;\sqrt{2})$ and $N(m/\sigma\,;\sqrt{2})$ respectively. From the corollary 3 of the Section I, it evidently follows that both v^2 and w^2 and Bessel variates with the parameters $\beta^2 = m^2/\sigma^2$ and $p = \frac{1}{2}$.

Hence the frequency function of the product xy becomes that for the difference of two Bessel variates v^2 and w^2. Now putting $p = \frac{1}{2}$ in the expression (3.10) the distribution of $u = |v^2-w^2|$ becomes

$$f(u)=e^{-\frac{1}{2}.m^2/\sigma^2}.\frac{1}{\pi^{\frac{1}{2}}}\sum_{r=0}^{\infty}\frac{\beta^{2r}u^{r}}{r\,!\Gamma(r+\frac{1}{2}).2^{2r}}.K_r(u). \tag{3.11}$$

But since $(2r)\,! = 2^{2r}r\,!\Gamma(r+\frac{1}{2})/\pi^{\frac{1}{2}}$

$$f(u)=\frac{e^{-\frac{1}{2}\beta^2}}{\pi}\sum_{r=0}^{\infty}\frac{\beta^{2r}u^{r}}{(2r)\,!}K_r(u). \tag{3.12}$$

The frequency function of the product xy where both x and y are distributed independently normally has been obtained by C. C. Craig (1936) by direct method.

Corollary 2: The distribution of the difference of two Gamma Variates.

Now as $\beta \to 0$ the limiting form of the distribution of u in (3.10) becomes

$$f(u)=\frac{u^{\frac{1}{2}(2p-1)}}{2^{\frac{1}{2}(2p-1)}}\cdot\frac{1}{\pi^{\frac{1}{2}}}\cdot\frac{1}{\Gamma(p)}.K_{p-\frac{1}{2}}(u) \tag{3.13}$$

where $u = x_1-x_2$, x_1 and x_2 being two independent Gamma variates. The distribution (3.13) has been already found by Karl Pearson, Stouffer and David (1932).

Corollary 3: The distribution of the difference of a Bessel Variate and a Gamma Variate.

Let us next proceed to find the distribution of $u = x_1 - x_2$ where x_1 is a Bessel variate and x_2 is a Gamma variate. Now putting $\beta_1 = \beta$ and $\beta_2 = 0$ in the expression in (3.8), the characteristic function for the distribution of u will be given by

$$\varphi_u(t) = e^{-\frac{1}{4}\beta^2}.\ \exp\left\{\frac{1}{4}\cdot\frac{\beta^2}{1-it}\right\}\cdot(1-it)^{-p_1}(1+it)^{-p_2}. \tag{3.14}$$

Next setting $1-it = -z/u$ the distribution function of u may be written as

$$f(u) = e^{-\frac{1}{4}\beta^2}\times\frac{e^{-u}u^{p_1-1}}{2^{p_2}}\times\frac{1}{2\pi i}\int_{-u-i\infty}^{-u+i\infty}\exp\left\{\frac{\beta^2u}{4(-z)}-z\right\}(-z)^{-p_1}\left(1+\frac{z}{2u}\right)^{-p_2}dz. \tag{3.15}$$

Now on expanding the exponent and integrating term by term as in (3.7) we shall get

$$f(u) = e^{-\frac{1}{4}\beta^2}\times\frac{u^{\frac{1}{2}(p_1+p_2)-1}}{2^{\frac{1}{2}(p_1+p_2)}}\times\sum_{r=0}^{\infty}\left(\frac{\beta}{2}\right)^{2r}\left(\frac{u}{2}\right)^{r/2}\cdot\frac{1}{r!\Gamma(p_1+r)}\cdot W_{\frac{1}{2}(p_1-p_2+r),\ \frac{1}{2}(p_1+p_2+r-1)}(2u) \tag{3.16}$$

For the particular case when $p_1 = p_2 = p$ the density function of u reduces to the form

$$f(u) = e^{-\frac{1}{4}\beta^2}.\frac{u^{p-1}}{2^p}\sum_{r=0}^{\infty}\left(\frac{\beta}{2}\right)^{2r}\cdot\left(\frac{u}{2}\right)^{r/2}\cdot\frac{1}{r!\Gamma(p_1+r)}\cdot W_{\frac{1}{2}r,\ \frac{1}{2}(2p-1+r)}.(2u). \tag{3.17}$$

Further when $p = \frac{1}{2}$ the expression (3.17) further reduces to the form

$$f(u) = e^{-\frac{1}{4}\beta^2}.\frac{1}{\sqrt{(2\pi u)}}\sum_{r=0}^{\infty}\frac{\beta^{2r}}{(2r)!}\cdot\left(\frac{u}{2}\right)^{r/2}\cdot W_{\frac{1}{2}r,\ \frac{1}{2}r}(2u). \tag{3.18}$$

Section IV *The distribution of a linear function of the independent Bessel Variates.*

Let us suppose that there are m independent Bessel variates $x_1, x_2 \ldots x_m$ with their distribution functions respectively given by

$$dF(x_j) = \text{const.}\ x_j^{\frac{1}{2}(p_j-1)}e^{-x_j}I_{p_j-1}(\beta_j\sqrt{x_j})dx_j \text{ for } j = 1, 2, \ldots m. \tag{4.1}$$

We shall next proceed to find the distribution of the linear function

$$u = \lambda(\lambda_1x_1+\lambda_1x_2+\ldots\ldots+\lambda_mx_m)$$

where we may assume without any loss of generality, that all the λ_j's are positive and greater than unity.

Now the characteristic function for the distribution of the variate x_j is given by

$$\varphi_j(t) = e^{-\frac{1}{4}\beta_j^2}\sum_{r=0}^{\infty}\frac{\beta_j^{2r}}{2^{2r}r!}(1-it)^{-(p_j+r)};\ j = 1, 2,\ldots m \tag{4.2}$$

which evidently follows from (1. 8) of Section I.

Hence the c. f. for the distribution of u/λ is given by

$$\varphi_{u/\lambda}(t) = \prod_{j=1}^{m} e^{-\frac{1}{4}\beta_j^2} \cdot \sum_{r=0}^{\infty} \frac{\beta_j^{2r}}{2^{2r} r\,!}(1-i\lambda_j t)^{-(p_j+k)}$$

$$= e^{-\frac{1}{4}\sum_{j=1}^{m}\beta_j^2} \sum_{r_1, r_2, \ldots, r_m=0}^{\infty} \frac{\beta_1^{2r_1}\beta_2^{2r_2}\ldots\ldots\beta_m^{2r_m}}{2^{2r_1+\ldots+2r_m} r_1\,!\ldots r_m\,!} \times$$

$$(1-i\lambda_1 t)^{-(p_1+r_1)}(1-i\lambda_2 t)^{-(p_2+r_2)}\ldots\ldots(1-i\lambda_m t)^{-(p_m+r_m)} \tag{4.3}$$

Now we may write $(1-i\lambda_j t) = \lambda_j(1-it)-(\lambda_j-1)$

$$= \lambda_j(1-it)\{1-(1-1/\lambda_j)(1-it)^{-1}\}$$

Hence $(1-i\lambda_j t)^{-\delta_j} = \lambda_j^{-\delta_j}(1-it)^{-\delta_j}\{1-(1-1/\lambda_j)(1-it)^{-1}\}^{-\delta_j}$

$$= \lambda_j^{-\delta_j}\sum_{r=0}^{\infty} \frac{\delta_j(\delta_j+1)\ldots.\,.(\delta_j+r-1)}{r\,!}(1-1/\lambda_j)^r(1-it)^{-(\delta_j+r} .$$

Now substituting there values in (4.3) the c.f. for the distribution of u/λ is given by

$$\varphi_{u/\lambda}(t) = \exp\{-\tfrac{1}{4}\sum_{j=1}^{m}\beta_j^2\} \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{\beta_1^{2r_1}\beta_2^{2r_2}\ldots\ldots\beta_m^{2r_m}}{2^{2r_1+\ldots+2r_m} r\,!\ldots r_m\,!} \sum_{r=0}^{\infty} a_r(1-it)^{-(p_1+\,..+p_m+r_1+.\,\,+r_m+r)} \tag{4.4}$$

where

$$a_r = \sum_{r_1+\,..+r_m=r} \{a_{1r_1}\;a_{2r_2}\ldots a_{mr_m}\}$$

and

$$a_{jr_j} = \lambda_j^{-(p_j+r_j)}\sum_{r=0}^{\infty}\frac{(p_j+r_j)\ldots(p_j+r_j+r-1)}{r\,!}\,(1-1/\lambda_j)^r, \quad \text{for } j = 1, 2, \ldots m.$$

Hence by the application of the Fourier Integral Theorem, the distribution of u/λ is given by

$$f(u/\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\,u/\lambda}\varphi_{u/\lambda}(t)\,dt$$

$$= \exp\left\{-\tfrac{1}{4}\sum_{j=1}^{m}\beta_j^2\right\} \sum_{r_1, r_2,\ldots r_m=0}^{\infty} \frac{\beta_1^{2r_1}\beta_2^{2r_2}.\,.\beta_m^{2r_m}}{2^{2r_1+\ldots+2r_m} r_1\,!\ldots r_m\,!}$$

$$\times \sum_{r=0}^{\infty} a_r \frac{e^{-u/\lambda}(u/\lambda)^{\Sigma_j(p_j+r_j)+r-1}}{\Gamma(p_1+p_2+\ldots+p_m+r_1+\ldots+r_m+r)} \tag{4.5}$$

since it is well-known that

$$\frac{1}{2\pi}\int_{\infty}^{\infty} e^{-itx}(1-it)^{-\alpha}dt = e^{-x}\frac{x^{\alpha-1}}{\Gamma(\alpha)}$$

Corollary 1: *The distribution of a linear function of Gamma Variates.*

On setting $\beta_j = 0$ for $j = 1, 2, \ldots m$ in (4. 5) the distribution of u/λ reduces to the form

$$f\left(\frac{u}{\lambda}\right) = \sum_{r=0}^{\infty} a_r.\, \frac{e^{-u/\lambda}(u/\lambda)^{p_1+\ldots+p_m+r-1}}{\Gamma(p_1+p_2+\ldots p_m+r)} \tag{4.6}$$

Corollary 2: The distribution of a positive definite non-central quadratic form in normally distributed variables

Let $x_1, x_2, \ldots x_n$ represent n independent normal variates distributed with means $a_1, a_2 \ldots a_n$ and a common variance σ^2. Let us proceed to find the distribution of $u = \sum_{ij=1}^{n} a_{ij}x_ix_j = \mathbf{x}A\mathbf{x}'$ where *rank* $A = m(\leq n)$ and further $A = ((aij))$ *real*, positive definite. Now we can always choose an orthogonal matrix C (*i. e.* $CC' = 1$) such that $CAC' = \Lambda$ where

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_m & & & \\ & & & & 0 & & \\ & & & & & 0. & \\ & & & & & & 0 \end{pmatrix}$$

$\lambda_1, \lambda_2 \ldots \lambda_m$ being the latent roots of the matrix A defined by the characteristic equation $|A - \lambda I| = 0$. Hence by using this wellknown result and choosing an orthogonal transformation defined by $\mathbf{x} = \mathbf{X}C$ where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ we must have $u = \mathbf{x}A\mathbf{x}' = \mathbf{X}\Lambda\mathbf{X}' = \lambda_1\mathbf{X}_1^2 + \lambda_2\mathbf{X}_2^2 + \ldots + \lambda_m\mathbf{X}_m^2$ where each $\lambda_i > 0$ ($i = 1, 2, \ldots m \leq n$). Now it evidently follows that $X_1, X_2 \ldots X_m$ are also independently distributed normally with a common variance σ^2.

Since $\mathbf{x} = \mathbf{X}C$, hence $E(\mathbf{x}) = E(\mathbf{X}) \cdot C$

and

$$E[\{\mathbf{x} - E(\mathbf{x})\}'\{\mathbf{x} - E(\mathbf{x})\}] = E[C'\{\mathbf{X} - E(\mathbf{X})\}'\{\mathbf{X} - E(\mathbf{X})\}C]$$

$$= C'E[\{\mathbf{X} - E(\mathbf{X})\}'\{\mathbf{X} - E(\mathbf{X})\}] \cdot C$$

Then

$$E[\{\mathbf{X} - E(\mathbf{X})\}'\{\mathbf{X} - E(\mathbf{X})\}]$$

$$= C \cdot F[\{\mathbf{x} - E(\mathbf{x})\}'\{x - E(\mathbf{x})\}] \cdot C'$$

$$= C \cdot (\sigma^2 \cdot I) \cdot C' = \sigma^2 \cdot I,$$ I representing the unit matrix.

Thus it follows evinently that $\mathbf{X}_1, \mathbf{X}_2 \ldots \mathbf{X}_n$ are also distributed independently normally with a common variance σ^2. Hence using the result of corrollary 3. of Section 1. each of the squares $\mathbf{X}_1^2, \mathbf{X}_2^2 \ldots \mathbf{X}_n^2$ is a Bessel variate.

Now we may write

$$u = \mathbf{x}A\mathbf{x}' = \lambda_1 X_1^2 + \lambda_2 X_2^2 + \cdots + \lambda_m X_m^2 = \lambda(\lambda_1' X_1^2 + \lambda_2' X_2^2 + \cdots + \lambda_m' X_m^2),$$

where each of $\lambda_i' \geq 1$ for $i = 1, 2 \ldots m(\leq n)$.

Hence the distribution of the form $\mathbf{x}A\mathbf{x}'$ will follow the distribution of u/λ as given in (4.5) of Section IV.

Section V. *The distribution of the ratio of two independent linear functions in Bessel Variates.*

Let us consider the distribution of

$$u = \frac{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m}{\lambda_1' x_1' + \lambda_2' x_2' + \cdots + \lambda_n' x_n'}$$

where the x's are independent Bessel variates and further $\lambda_1, \lambda_2 \ldots \lambda_m$ and $\lambda'_1, \lambda'_2 \ldots \lambda'_n$ are all positive and greater than unity. Further Let

$$dF(x_j) = \text{const. } x_j^{\frac{1}{2}(p_j-1)} e^{-x_j} I_{p_j-1}(\beta\, x_j^{\frac{1}{2}}) dx_j, \quad (j = 1, 2, \ldots m)$$

$$dF(x_k') = \text{const. } x_k'^{\frac{1}{2}(p'_k-1)} e^{-x'_k} I_{p'_k-1}(\beta_k' x_k'^{\frac{1}{2}}) dx_k', \; (k = 1, 2, \ldots n). \tag{5.1}$$

Now, for simplicity in notation, let us write

$$u = y/z, \quad \text{where } y = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m$$

$$z = \lambda_1' x_1' + \lambda_2' x_2' + \ldots + \lambda_n' x_n'$$

Using the relation (4.5) of Section IV, the distributions of y and z will be given by

$$dF(y) = \exp\{-\tfrac{1}{4}(\beta_1^2 + \beta_2^2 + \cdots + \beta_m^2)\} \cdot \sum_{r_1, r_2 \ldots r_m = 0}^{\infty} \frac{\beta_1^{2r_1} \beta_2^{2r_2} \ldots \beta_m^{2r_m}}{2^{2r_1+2r_2+ \ldots +2r_m} r_1! \ldots r_m!}$$

$$\times \sum_{r=0}^{\infty} a_r \cdot \frac{e^{-y} . y^{p_1+ \ldots +p_m+r_1+ \ldots +r_m+r-1}}{\Gamma(p_1 + \cdot + p_m + r_1 + \cdot + r_m + r)}$$

$$dF(z) = \exp\{-\tfrac{1}{4}(\beta_1'^2 + \beta_2'^2 + \cdots + \beta_n'^2)\} \cdot \sum_{s_1, s_2, \ldots s_n = 0}^{\infty} \frac{\beta_1'^{2s_1} \beta_2'^{2s_2} \ldots \beta_n'^{2s_n}}{2^{2s_1+ \ldots +2s_n} s_1! \ldots s_n!}$$

$$\times \sum_{s=0}^{\infty} b_s \cdot \frac{e^{-z} . z^{p_1'+p_2'+ \ldots +p_n'+s_1+ \ldots +s_n+s-1}}{\Gamma(p_1' + p_2' + \ldots + p_n' + s_1 + \ldots + s_n + s)} \tag{5.2}$$

Now first of all, we shall derive the distribution of

$$w = \log u = \log y - \log z.$$

Then the characteristic function of the distribution of $w = \log u$ will be given by

$$\varphi_w(t) = \int_0^\infty \int_0^\infty e^{iwt} dF(y) dF(z)$$

$$= \exp\left\{-\frac{1}{4}\left(\sum_{j=1}^{m} \beta_j^2 + \sum_{k=1}^{n} \beta_k'^2\right)\right\} \cdot \sum_{r_1, r_2 \ldots r_m = 0}^{\infty} \frac{\beta_1^{2r_1} \beta_2^{2r_2} \ldots \beta_m^{2r_m}}{2^{2r_1+2r_2+ \ldots +2r_m} r_1! \ldots r_m!}$$

$$\cdot \sum_{s_1, s_2, \ldots s_n = 0}^{\infty} \frac{\beta_1'^{2s_1} \beta_2'^{2s_2} \ldots \beta_n'^{2s_n}}{2^{2s_1+2s_2+ \ldots +2s_n} s_1! \ldots s_n!}$$

$$\times \sum_{r, s = 0}^{\infty} \frac{a_r b_s}{\Gamma(p_1 + \cdot + p_m + r_1 + \cdot + r_m + r) . \Gamma(p_1' + \cdot + p_n' + s_1 + \cdot + s_n + s)}$$

$$\times \int_0^\infty y^{it} e^{-y} y^{p_1+ \ldots +p_m+ \ldots +r_m+r-1} dy \times \int_0^\infty z^{-it} e^{-z} z^{p_1'+ \ldots +p_n'+ \ldots +s_n+s-1} dz. \tag{5.3}$$

Now proceeding exactly in the same lines as in Section II the frequency function of w becomes

$$f(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwt} \varphi_w(t) dt$$

and finally the distribution $u = e^{w}$ becomes after some simplifications,

$$f(u) = \exp\left\{-\frac{1}{4}\left(\sum_{j=1}^{m}\beta_j^2 + \sum_{k=1}^{n}\beta_k'^2\right)\right\}$$

$$\cdot \sum_{r_1, r_2, \ldots r_m=0}^{\infty} \frac{\beta_1^{2r_1} \ldots \beta_m^{2r_m}}{2^{2r_1+\ldots+2r_m} r_1! \ldots r_m!} \cdot \sum_{s_1, s_2, \ldots s_n=0}^{\infty} \frac{\beta_1'^{2s_1} \ldots \beta_n'^{2s_n}}{2^{2s_1+\ldots+2s_n} s_1! \ldots s_n!}$$

$$\times \sum_{r, s=0}^{\infty} \frac{a_r b_s}{B(P, Q)} \cdot \frac{u^{P-1}}{(1+u)^{P+Q}} \qquad (5\ 4)$$

where

$$P = p_1 + p_2 + \cdots + p_m + r_1 + \cdots + r_m + r$$

$$Q = p_1' + p_2' + \cdots + p_n' + s_1 + \cdots + s_n + s$$

where a_r and b_s may be expressed as in Section IV.

Corrollary 1: The distribution of the ratio of two independent linear functions of Gamma Variates.

Now on putting $\beta_j = 0$ for $j = 1, 2, 3, \ldots m$ and also $\beta_k' = 0$ for $k = 1, 2, 3, \ldots n$ in (5.4), the frequency function for u that is, the ratio of two independent linear functions of Gamma Variates becomes

$$f(u) = \sum_{r, s=0}^{\infty} \frac{a_r b_s}{B(p_1+p_2+\cdots+p_m+r,\ p_1'+\cdots+p_n'+s)} \cdot \frac{u^{p_1+p_2+\ldots+p_m+r-1}}{(1+u)^{p_1+\ldots+p_m+p_1'+\ldots+p_n'+r+s}}. \qquad (5.5)$$

Corrollary 2: The distribution of the ratio of two positive definite non-central quadratic forms in independent normal variates.

Let $x_1, x_2 \ldots x_k$ be k independent normal variates distributed with means $\alpha_1, \alpha_2 \ldots \alpha_k$ and a common variance σ^2.

Let us now proceed to find out the distribution of the ratio $u = \mathbf{x}A\mathbf{x}'/\mathbf{x}B\mathbf{x}'$ where $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ and rank $A = m(\leq k)$ and rank $B = n(\leq k)$ both A and B being real, positive definite. By applying proper orthogonal transformations the numerator and the denominator of the above ratio can be expressed as linear functions of m and n Bessel Variates respectively, with the corresponding latent roots as the coefficients. Hence, following exactlythe same lines as in Section IV and Section V, the distribution of the ratio $(\mathbf{x}A\mathbf{x})/\mathbf{x}B\mathbf{x}')$ will be of the same form as given in (5.4).

Summary. In the present paper a unified approach towards solving some distribution problems connected with Bessel variates is made by applying the method of characteristic functions. In section I the reproductive property of the Bessel variates (that is, the distribution of the sum of a number of independent Bessel variates is also a Bessel variate) is discussed. In Sections II and III the distributions of the ratio and the difference of two independent Bessel variates are derived. Further generalizations of the results are made in Sections IV and V which respectively deal with the linear function of the Bessel variates and the ratio of two such independent linear functions. In each of the sections a number of important corrollaries are discussed.

In conclusion the author wishes to express his thanks to Dr. P. K. Bose for his keen interest in this work and also for some valuable suggestions.

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CALCUTTA.

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CALCUTTA MATHEMATICAL SOCIETY.

Report of the Council for the year 1953 to the Annual General Meeting of the Society

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the general concerns of the Society for the year 1953 as required by the provisions of Rule 25.

The Council: The Council of the Society for the year 1953 consisting of the officers and other members elected at the last Annual General Meeting and co-opted thereafter, together with the Editorial Secretary, was constituted as follows :

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Dr. S. S. Bhatnagar, F. R. S.

General: The various activities of the Society have been carried on throughout the year in much the same form as in the past few years. The Council has the pleasure to report that on an invitation from the Society Sir Richard Southwell, F. R. S., Professor of Engineering Mathematics in the University of London delivered a lecture on some problems in the theory of Elasticity at a meeting of the Society in January, 1953, when he came to India to attend the Indian Science Congress Session. The Society also arranged another lecture on "Atomic Collisions" by Prof. H. S. W. Massey, F. R. S., Goldsmid Professor of Mathematics in the University of London, when he had occasion to halt at Calcutta on his way back to London from Japan in October, 1953. The Council conveys its grateful thanks to these distinguished Scientists for their stimulating lectures.

Membership: The Council reports with pleasure that 11 new names have been entered in the Society's list of members during the year under review.

Meetings during 1953: The Council held six meetings during the year and there were five ordinary general meetings devoted to the reading of original papers communicated to the Society for publication in its Bulletin.

Publications: During the year under review the Society has published five numbers of the Bulletin, namely, Vol. 44, Nos. 2-4 and Vol. 45 Nos. 1-2.

The Council desires to report that the appointment of a Compositor in the University Press to work exclusively in printing the Society's Bulletin, has expedited the publications to a large degree and it may reasonably be expected that all arrears in publications will be wiped off before the current year is out. The Council takes the opportunity to record here the Society's indebtedness to the authorities of the Calcutta University for printing the Bulletin free of charge and the Officers and members of the staff of the Calcutta Uniuersity Press for their valued services.

Exchange of Publications: The transmission of the Society's publications to various countries in the world has been carried on regularly during the year and some new exchange relations have also been established.

Library: It is gratifying to note that there has been no break during the year under review in the run of periodicals which the Society has been subscribing for the last few years, out of grants received from the Central and State Governments. These grants have also enabled the Society to add to its library a collection of some recent standard books on various branches of Pure and Applied Mathematics.

Finance: The annual accounts of the Society for the year under review have been presented to the Council in the standardised form by the auditors Dr. A. C. Choudhury and Mr. N. N. Ghosh. The Council gratefully acknowledges its indebtedness to them for their honorary services. The Society received the following grants during the year under review:—

(*i*)	Government of West Bengal ...	Rs. 1,000/-
(*ii*)	National Institute of Sciences of India	Rs. 500/-

and the Council takes this opportunity to offer its grateful thanks to the Government of West Bengal and the National Institute of Sciences for these grants.

CALCUTTA MATHEMETICAL SOCIETY

RECEIPT AND DISBURSEMENTS ACCOUNTS OF THE CALCUTTA MATHEMATICAL SOCIETY FOR THE YEAR ENDING 31ST DECEMBER, 1953.

Receipts

		Rs. as. p.	Rs. as. p.
1. Opening Balance			
(a) With Secretary			
(i) In cash ...	12 4 3		
(ii) In stamps ...	4 7 3		
		16 11 6	
(b) Balance at Banks			
(i) Imperial Bank of India	4,246 14 3		
(ii) Do (K. K. G, P. Fund)	635 9 8		
(iii) United Bank of India	1,435 11 9		
Do (in suspense)	12 0 0		
(iv) Postal Savings Bank	893 13 0		
		7,224 0 8	
(c) G. P. Notes (General Fund) (Face value Rs. 6,000) ...		5,663 11 6	
(d) G. P. Notes (K. K. G P. Fund) (Face value Rs. 2,000) ...		1,937 7 9	
			14,841 15 5
2. Membership subscription ...	852 10 0		
3. Admission fees ...	80 0 0		
4. Sale proceeds ...	1,471 2 3		
Do (in suspense) ...	88 12 0		
		2,492 8 3	
5. Grants			
(i) Govt. of West Bengal ...	1,000 0 0		
(ii) Nat. Inst. of Sciences ...	500 0 0		
		1,500 0 0	
6. Interest			
(a) G. P. Notes (General Fund)	180 0 0		
(b) Do (K. K G. P. Fund) ...	60 0 0		
(c) Postal Savings Bank ...	13 0 0		
		253 0 0	
			4,245 8 3
TOTAL ... Rs.			19,087 7 8.

Disbursements

		Rs. as. p.	Rs. as. p.
1. Establishment ...		1,028 7 9	
2. Meetings ...		122 6 0	
3. Books & Journals (including binding) ...		1,769 5 6	
5. Bulletins			
(a) Compositor's Salary	900 0 0		
(b) Papers, Blocks Types etc.	729 0 6		
(c) Postage	374 14 0		
		2,003 14 6	
5. Printing & Stationery ...		46 11 0	
6. Postage (general) ...		156 9 3	
7. Bank charges ..		19 3 0	
8. Miscellaneous (including conveyance charges)		100 5 3	
			5,246 14 3
Closing Balance			
(a) With Secretary			
(i) In cash	2 0 3		
(ii) In stamps	3 0 0		
		5 0 3	
(b) Balance at Banks			
(i) Imperial Bank of India	3,625 10 0		
(ii) Do (K. K. G. P. Fund)	695 1 8		
(iii) United Bank of India	888 1 3		
Do (in suspense)	88 12 0		
(iv) Postal Savings Bank	936 13 0		
		6,234 5 11	
(c) G. P. Notes (General Fund) (Face value Rs. 6,000) ...		5,663 11 6	
(d) G. P Notes (K. K. G. P. Fund) (Face value Rs. 2.000) ...		1,937 7 9	
			13.840 9 5
TOTAL ... Rs.			19.087 7 8

To

The Members of the Calcutta Mathematical Society,

We have examined the above Balance Sheet with the Books and Vouchers relating thereto and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

A. C. CHOUDHURY,
N. N. GHOSH.
Auditors.

BULLETIN

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EXTENSION OF A THEOREM OF GLAISHER AND SOME RELATED RESULTS

By

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1. The well-known formula of Wolstenholme

$$\sum_{r=1}^{p-1}\frac{1}{r}\equiv 0 \pmod{p^2}$$

where p is a prime >3, has been generalized by Leudesdorf [for references see Hardy and Wright (1938), Chap. VII]; this in turn has been generalized by Rama Rao (1937). In this note we consider the numbers $A_r^{(k)}$ defined by

$$\prod_{r=1}^{p-1}(x-kp-r) = x^{p-1}-A_r^{(k)}x^{p-2}+\dots+A_{p-1}^{(k)}, \tag{1. 1}$$

where k is an arbitrary integer and $p>3$. For the numbers $A_r = A_r^{(o)}$ Glaisher [Glaisher (1900*b*) § 9]

$$\left.\begin{aligned} \frac{1}{p}A_{2r} &\equiv -\frac{1}{2r}B_{2r} \pmod{p} \\ \frac{1}{p^2}A_{2r+1} &\equiv \frac{2r+1}{4r}B_{2r} \pmod{p}, \end{aligned}\right\} \tag{1. 2}$$

where $1\leq r\leq \frac{1}{2}(p-3)$ and B_{2r} denotes the Bernoulli number in the even-suffix notation.

If in (1. 1) we take $x=(2k+1)p$, we get

$$A_{p-2}^{(k)}-(2k+1)pA_{p-3}^{(k)}+(2k+1)^2p^2A_{p-4}^{(k)}-\dots=0, \tag{1. 3}$$

from which it follows that

$$A_{p-2}^{(k)}\equiv 0 \pmod{(2k+1)p^2}$$

and

$$A_{p-2}^{(k)}\equiv(2k+1)pA_{p-3}^{(k)} \pmod{(2k+1)^2p^3}; \tag{1. 4}$$

indeed (1 5) holds (mod $(2k+1)^2p^4$) for $p>5$. In particular for properly choosen k, $A_{p-2}^{(k)}$ is divisible by arbitrarily high powers of p.

2. In the notation of Nörlund [Nörlund (1924), Chap. V] put

$$\left(\frac{t}{e^t-1}\right)^k e^{xt}=\sum_{m=0}^{\infty}\frac{t^m}{m!}B_m^{(k)}(x), \quad B_m^{(k)}=B_m^{(k)}(0).$$

Then
$$B_k^{(k+1)}(x) = (x-1)(x-2)\dots(x-k)$$
and
$$B_m^{(k)}(x+y) = \sum_{s=0}^{m}\binom{m}{s}x^{m-s}B_s^{(k)}(y). \tag{2. 1}$$

In this notation the left member of (1. 1) is $B_{p-1}^{(p)}(x-kp)$, and
$$(-1)^s A_s^{(k)} = \binom{p-1}{s}B_s^{(p)}(-kp). \tag{2. 2}$$

Now by (2. 1)
$$B_{2r+1}^{(p)}(-kp) = B_{2r+1}^{(p)} - (2r+1)kpB_{2r}^{(p)} + \binom{2r+1}{2}k^2p^2B_{2r-1}^{(p)} - \dots,$$
so that
$$-\binom{p-1}{2r+1}B_{2r+1}^{(p)}(-kp) = -\binom{p-1}{2r+1}B_{2m+1}^{(p)} + kp(p-2r-1)\binom{p-1}{2r}B_{2r}^{(p)}$$
$$-\tfrac{1}{2}k^2p^2(p-2r-1)\ (p-2r)\binom{p-1}{2r-1}B_{2r-1}^{(p)} + \dots$$
$$= A_{2r+1} + kp(p-2r-1)A_{2r} + \tfrac{1}{2}k^2p^2(p-2r-1)(p-2r)A_{2r-1} + \dots \tag{2. 3}$$

By another formula of Glaisher [Glaisher (1900*a*), § 19]
$$A_{2r+1} \equiv \tfrac{1}{2}p(p-2r-1)A_{2r} \pmod{p^4} \qquad (p > 5,\ r > 1);$$
thus (2. 3) implies
$$A_{2r+1}^{(k)} \equiv \tfrac{1}{2}(2k+1)p(p-2r-1)A_{2r} \pmod{p^4} \qquad (p > 5,\ r > 1). \tag{2. 4}$$

Using (1. 2) we get
$$A_{2r+1}^{(k)} \equiv \frac{(2k+1)(2r+1)}{4r}p^2B_{2r} \pmod{p^3}, \tag{2. 5}$$
which holds for $r \geq 1$, $p > 3$.

In the second place
$$B_{2r}^{(p)}(-kp) = B_{2r}^{(p)} - 2rkpB_{2r-1}^{(p)} + \dots$$
$$\equiv B_{2r}^{(p)} \pmod{p^3} \qquad (r > 1).$$

Consequently by (2. 2) and (1. 2)
$$A_{2r}^{(k)} \equiv -\frac{1}{2r}pB_{2r} \pmod{p^2}, \tag{2. 6}$$
which holds for $r \geq 1$, $p > 3$.

Thus (2. 5) and (2. 6) furnish a generalisation of (1. 2).

3. Let
$$S_{k,r} = \sum_{s=1}^{p-1}\frac{1}{(kp+s)^r}.$$

Then we have

$$2S_{k,2r+1}=\sum_{s=1}^{p=1}\frac{1}{(kp+s)^{2r+1}}+\sum_{s=1}^{p=1}\frac{1}{((k+1)p-s)^{2r+1}}.$$

Since $(kp+s)^{2r+1}+((k+1)p-s)^{2r+1}$ is divisible by $(2k+1)p$, it follows that

$$S_{k,2r+1}\equiv 0 \pmod{(2k+1)p} \qquad (p\geq 3). \tag{3. 1}$$

On the other hand if A'_r denotes the r-th elementary symmetric function of the number $(kp+1)^{-1}$, ..., $(kp+p-1)^{-1}$, then by Newton's formula

$$S_{k,r}+A'_1S_{k,r-1}+\dots+A'_{r-1}S_{k,1}+rA'_r=0 \qquad (r=1,\dots,\ p-1). \tag{3. 2}$$

For $r=1$, (3. 2) implies $A'_1\equiv 0 \pmod{(2k+1)\,p}$; for $r=3$, we use (3. 1) and find that $A'_3\equiv 0 \pmod{(2k+1)\,p}$. It is now clear how we can prove generally that $A'_{2r+1}\equiv 0 \pmod{(2k+1)\,p}$. Since

$$A_r^{(k)}=(kp+1)\dots(kp+p-1)A'_{p-1-r},$$

it follows that

$$A^{(k)}_{2r+1}\equiv 0 \pmod{(2k+1)p}.$$

In particular (3. 3) implies that for properly chosen k, $A^{(k)}_{2r+1}$ is divisible by arbitrarily high powers of p.

4. Jacobsthal and Ljunggren (1947) have proved the formula $(p>3)$

$$\binom{p^{r+1}}{p^r}-\binom{p^r}{p^{r-1}}\equiv\tfrac{1}{3}p^{3r+2}B_{p-3} \pmod{p^{3r+3}}, \tag{4. 1}$$

as well a more precise results. In particular (4. 1) implies

$$\binom{p^{r+1}}{p^r}-\binom{p^r}{p^{r-1}}\equiv 0 \pmod{p^{3r+2}}.$$

If we put $H^r=\binom{np^r}{p^r}$, then it is easily verified that

$$H_r-H_{r-1}=H_{r-1}\sum_{s>1}(-1)^s k_s(np^r)^s, \tag{4. 3}$$

where k_s is s-th elementary symmetric function of the numbers t^{-1}, $1\leq t<p^r$, not $p\mid i$ Using [Rama Rao (1937), Theorem 3] and the fact that $H_{r-1}\equiv 0 \pmod{n}$ it follows at once from (4. 3) that

$$H_r-H_{r-1}\equiv 0 \pmod{n^2p^{3r}} \qquad (p>3). \tag{4. 4}$$

For $n=p$, (4. 4) reduces to (4. 2).

To obtain a more precise result we require [*cf.* Ljunggren (1947) p. 8]

$$S_1=\sum_{\substack{s=1\\ \text{not } p\mid s}}^{p^r-1}\frac{1}{s}\equiv\sum_{s=1}^{p^r-1}s^{N-1} \qquad (N=p^{3r-1}(p-1))$$

$$\equiv\frac{1}{N}(B_N(p^r)-B_N) \qquad \pmod{p^{3r}},$$

where $B_m(x)$ is the Bernoulli polynomial of order 1. It follows that

$$S_1 \equiv -\tfrac{1}{2}B_{N-2}p^{2r} \qquad (\mathrm{mod}\ p^{3r}) \tag{4. 5}$$

Similarly we have

$$S_2 = \sum_{\substack{s=1 \\ \text{not } p \mid s}}^{p^r-1} \frac{1}{s^2} \equiv B_{N-2}p^r \qquad (\mathrm{mod}\ p^{3r}). \tag{4. 6}$$

Substituting from (4. 5) and (4. 6) in (4. 3) we get

$$H_r - H_{r-1} \equiv -\tfrac{1}{2}n(n-1)p^{3r}H_{r-1}B_{N-2} \qquad (\mathrm{mod}\ n^2p^{4r}). \tag{4. 7}$$

This result can be simplified by employing Kummer's congruence [Frobenius (1910), p. 842].

$$\frac{B_{m+t}}{m+t} \equiv \frac{B_m}{m} \qquad (\mathrm{mod}\ p^e)$$

for $p^{e-1}(p-1) \mid t$, not $p-1 \mid m$, $m > e$. Thus for example

$$\frac{B_{N-2}}{N-2} \equiv \frac{B_{M-2}}{M-2} \qquad (\mathrm{mod}\ p^{r+1}) \qquad (M = p^r(p-1)) \tag{4. 8}$$

and

$$\frac{B_{N-2}}{N-2} \equiv \frac{B_{p-3}}{p-3} \qquad (\mathrm{mod}\ p). \tag{4. 9}$$

Using (4. 8), (4. 7) becomes

$$H_r - H_{r-1} \equiv -\tfrac{1}{2}n(n-1)p^{3r}H_{r-1}B_{M-2} \qquad (\mathrm{mod}\ (n^2p^{4r},\ np^{4r+1})), \tag{4. 10}$$

while by means of (4. 9) we get

$$H_r - H_{r-1} \equiv -\tfrac{1}{3}n(n-1)p^{3r}H_{r-1}B_{p-3} \qquad (\mathrm{mod}\ n^2p^{3r+1}). \tag{4. 11}$$

In particular when $p \mid n$, (4. 11) implies

$$H_r - H_{r-1} \equiv \tfrac{1}{3}\, n^2p^{3r}B_{p-3} \qquad (\mathrm{mod}\ p^{3r+3}), \tag{4. 12}$$

which includes (4. 1) as a special case.

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RESISTANCE ON AN INFINITE CYLINDER DUE TO TWO-DIMENSIONAL MOTION PAST THE CYLINDER, OF A FLUID HAVING UNIFORM VORTICITY

By

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1. It has been shown in a previous paper (De, 1953) that Blasius Theorem can be extended to calculate the thrust due to liquid motion past a cylinder when the liquid has vorticity. There it was proved that thrust and the moment are given by

$$X - iY = \tfrac{1}{2} i\varrho \int_{C} (u - iv)^2 dz \qquad (1)$$

$$\text{Moment} = -\tfrac{1}{2}\varrho \times Real \int z(u - iv)^2 dz \qquad (2)$$

In this paper these formulae will be used to calculate the resistance of a cylinder whose cross-section is bounded by the curve $\eta = \eta_o > 0$ in the co-ordinate system given by the equation defined by

$$z = nae^{-i\zeta} + be^{in\zeta}, \quad b \leq a \qquad (3)$$

where $z = x + iy$, $\zeta = \xi + i\eta$.

The integrations are comparatively easy due to the fact that the value of $u - iv$ on the boundary can be expressed as an analytic function of ζ

2. First we consider the case when the motion at infinity is a shearing motion parallel to a straight line OX' making an angle α with x-axis.

Let the shearing motion at infinity be given by

$$u' = -U - 2\omega y', \ v' = 0 \text{ along } OX' \text{ and } OY'.$$

where 2ω is the constant vorticity of the fluid. The stream function of the undisturbed motion is then

$$\begin{aligned}
\psi_o &= Uy' + \omega y'^2 \\
&= \tfrac{1}{2}\omega[n^2a^2e^{2\eta} + b^2e^{-2n\eta} + 2nabe^{-(n-1)\eta}\cos(n+1)\xi] \\
&\quad - \tfrac{1}{2}\omega[n^2a^2e^{2\eta}\cos 2\xi + b^2e^{-2n\eta}\cos 2n\xi + 2nabe^{-(n-1)\eta}\cos(n-1)\xi]\cos 2\alpha \\
&\quad - \tfrac{1}{2}\omega[b^2e^{-2n\eta}\sin 2n\xi - n^2a^2e^{2\eta}\sin 2\xi + 2nabe^{-(n-1)\eta}\sin(n-1)\xi]\sin 2\alpha \\
&\quad + U(be^{-n\eta}\sin n\xi - nae^{\eta}\sin\xi)\cos\alpha - U(nae^{\eta}\cos\xi + be^{-n\eta}\cos n\xi)\sin\alpha.
\end{aligned} \qquad (4)$$

If $\psi = \psi_o + \psi_2$ be the stream function for the disturbed motion, ψ_2 satisfies the equation $\nabla_1{}^2\psi_2 = 0$, and on the boundary $\eta = \eta_o$, $\psi_o + \psi_2$ is constant. We see at once that

$$\psi_2 = -nab e^{2\eta_0-(n+1)\eta}\cos(n+1)\xi + \tfrac{1}{2}\omega n^2a^2\cos 2\alpha e^{4\eta_0-2\eta}\cos 2\xi + \tfrac{1}{2}\omega b^2\cos 2\alpha e^{-2n\eta}\cos 2n\xi$$
$$+\omega nab\cos 2\alpha e^{-(n-1)\eta}\cos(n-1)\xi - \tfrac{1}{2}\omega n^2a^2\sin 2\alpha e^{4\eta_0-2\eta}\sin 2\xi + \tfrac{1}{2}\omega b^2\sin 2\alpha e^{-2n\eta}\sin 2\xi$$
$$+\omega nab\sin 2\alpha e^{-(n-1)\eta}\sin(n-1)\xi - Ub\cos\alpha e^{-n\eta}\sin n\xi + Ub\sin\alpha e^{-n\eta}\cos n\xi$$
$$+ Una\cos\alpha e^{2\eta_0-\eta}\sin\xi + Unae^{2\eta_0-\eta}\cos\xi\sin\alpha$$

It is therefore the imaginary part of

$$f_2 = Ae^{i\zeta} + Be^{i2\zeta} + Ce^{i(n-1)\zeta} + De^{in\zeta} + Ee^{i(n+1)\zeta} + Fe^{i2n\zeta}.$$

where

$$\left.\begin{aligned} A &= naUe^{2\eta_0}\cos\alpha + inaUe^{2\eta_0}\sin\alpha = naUe^{2\eta_0+i\alpha}\\ B &= -\tfrac{1}{2}\omega n^2a^2e^{4\eta_0}\sin 2\alpha + \tfrac{1}{2}i\omega n^2a^2e^{4\eta_0}\cos 2\alpha = \tfrac{1}{2}i\omega n^2a^2e^{4\eta_0+i2\alpha}\\ C &= \omega nab\sin 2\alpha + i\omega nab\cos 2\alpha = i\omega nabe^{-i2\alpha}\\ D &= -Ub\cos\alpha + iUb\sin\alpha = -Ube^{-i\alpha}\\ E &= -i\omega nabe^{2\eta_0}\\ F &= \tfrac{1}{2}\omega b^2\sin 2\alpha + \tfrac{1}{2}i\omega b^2\cos 2\alpha = \tfrac{1}{2}i\omega b^2e^{-i2\alpha} \end{aligned}\right\} \tag{5}$$

Since $z' = x'+iy' = ze^{-i\alpha}$, we can write

$\psi_0 = \tfrac{1}{2}\omega(x^2+y^2)+\psi_1$ where ψ_1 is the imaginary part of

$f_1 = Uze^{-i\alpha} - \tfrac{1}{2}i\omega z^2e^{-2i\alpha}.$

The complex velocity at any point is given by

$$u-iv = -i\omega\bar{z} - Ue^{-i\alpha} + i\omega ze^{-i2\alpha}$$
$$-i\Big[Ae^{i\zeta} + 2Be^{i2\zeta} + (n-1)Ce^{i(n-1)\zeta} + nDe^{in\zeta} + (n+1)Ee^{i(n+1)\zeta} + 2nFe^{i2n\zeta}\Big]\frac{d\zeta}{dz}$$

Since on $\eta = \eta_0$, $\bar{\zeta} = \zeta - -2i\eta_0$, we have on $\eta = \eta_0$,

$$\bar{z} = nae^{i\bar{\zeta}} + be^{-in\bar{\zeta}} = nae^{i\zeta+2\eta_0} + be^{-in\zeta-2n\eta_0}.$$

Therefore, on $\eta = \eta_0$

$$u-iv = -i\omega[nae^{i\zeta+2\eta_0} + be^{-in\zeta-2n\eta_0}] - Ue^{-i\alpha} + i\omega(nae^{-i\zeta}+be^{in\zeta})e^{-i2\alpha}$$
$$-i\Big[Ae^{i\zeta} + 2Be^{i2\zeta} + (n-1)Ce^{i(n-1)\zeta} + nDe^{in\zeta} + (n+1)Ee^{i(n+1)\zeta} + 2nFe^{i2n\zeta}\Big]\frac{d\zeta}{dz} \tag{6}$$

If

$$M = -Ue^{-i\alpha} + i\omega e^{-i2\alpha}(nae^{-i\zeta}+be^{in\zeta}) - i\omega(nae^{i\zeta+2\eta_0}+be^{-in\zeta-2n\eta_0})$$
$$N = -i[Ae^{i\zeta} + 2Be^{i2\zeta} + (n-1)Ce^{i(n-1)\zeta} + nDe^{in\zeta} + (n+1)Ee^{i(n+1)\zeta} + 2nFe^{i2n\zeta}]$$

then

$$X-iY = \tfrac{1}{2}i\varrho\int_C (u-iv)^2\frac{dz}{d\zeta}\,d\zeta = \tfrac{1}{2}i\varrho\int_C\Big(M^2\frac{dz}{d\zeta} + N^2\frac{d\zeta}{dz} + 2MN\Big)d\zeta \tag{7}$$

and

$$\text{Moment} = -\tfrac{1}{2}\varrho.\ \mathit{Real}\int_C z\Big(M^2\frac{dz}{d\zeta} + N^2\frac{d\zeta}{dz} + 2MN\Big)d\zeta \tag{8}$$

the integrations being taken round the boundary.

From (3) we have

$$\frac{dz}{d\zeta} = in(be^{in\zeta} - ae^{-i\zeta})$$

so that on $\eta = \eta_0$,

$$\frac{d\zeta}{dz} = \frac{1}{dz/d\zeta} = -\frac{e^{i\zeta}}{ina}\left[1 + \frac{b}{a}\,e^{i(n+1)\zeta} + \frac{b^2}{a^2}\,e^{2i(n+1)\zeta} + \ldots\ldots\right]$$

the expansion being valid, since $b \leq a$ and $\eta_0 > 0$

On $\eta = \eta_0$

$$\int_c e^{in\zeta} d\zeta = 0 \text{ when } n \neq 0 \text{ and } = 2\pi \text{ when } n = 0$$

Hence to calculate the integrals in (6) and (7) we have only to consider the constant terms in the integrands. We have

$$\int_c N^2 \frac{d\zeta}{dz} d\zeta = 0, \quad \int_c N^2 z \frac{d\zeta}{dz} d\zeta = 0$$

as there is no constant terms in the integrands.

The constant term in $M^2 \dfrac{dz}{d\zeta}$ is

$$-2n\omega U e^{-i\alpha}(b^2 e^{-2n\eta_0} - na^2 e^{2\eta_0})$$

The constant term in $2MN$ is

$$2n\omega U e^{-i\alpha}(b^2 e^{-2n\eta_0} + na^2 e^{2\eta_0})$$

when $n \neq 1$ or 2. It is

$$2\omega U e^{-i\alpha}(b^2 e^{-2\eta_0} + a^2 e^{2\eta_0}) - 2\omega U ab(e^{-3i\alpha} + e^{i\alpha})$$

when $n = 1$, and

$$4\omega U e^{-i\alpha}(b^2 e^{-4\eta_0} + 2a^2 e^{2\eta_0}) + 8i\omega^2 a^2 b(e^{-4i\alpha} - e^{2i\alpha})$$

when $n = 2$.

Therefore, when $n \neq 1$ or 2,

$$X - iY = 4n^2 \pi i \varrho \omega U a^2 e^{2\eta_0 - i\alpha}$$

or

$$X = 4n^2 \pi \varrho \omega U a^2 e^{2\eta_0} \sin\alpha, \quad Y = -4n^2 \pi \varrho \omega U a^2 e^{2\eta_0} \cos\alpha$$

When $n = 1$,

$$X - iY = 4\pi i \varrho \omega U a^2 e^{2\eta_0 - i\alpha} - 2\pi i \varrho \omega U ab(e^{-3i\alpha} + e^{i\alpha})$$

or

$$X = 4\pi \varrho \omega U \sin\alpha(a^2 e^{2\eta_0} - ab\cos 2\alpha)$$

$$Y = -4\pi \varrho \omega U \cos\alpha(a^2 e^{2\eta_0} - ab\cos 2\alpha)$$

When $n = 2$,

$$X - iY = 16\pi i \varrho \omega U a^2 e^{2\eta_0 - i\alpha} + 8\pi i \varrho \omega^2 a^2 b(e^{-4i\alpha} - e^{2i\alpha})$$

or

$$X = 16\pi \varrho \omega a^2 \sin\alpha(U e^{2\eta_0} + \omega b \sin 3\alpha)$$

$$Y = -16\pi \varrho \omega a^2 \cos\alpha(U e^{2\eta_0} + \omega b \sin 3\alpha)$$

For the calculation of the moment, we observe that the real part of the constant term is $M^2 z \dfrac{dz}{d\zeta}$ is zero except when $n = 1$. It is

$$2\omega^2 ab \sin 2\alpha(b^2 e^{-2\eta_0} - a^2 e^{2\eta_0})$$

when $n = 1$.

The real part of the constant term in $2MNz$ is zero except when $n = 1, 2$ or 3. It is

$$-2U^2ab \sin 2\alpha + 2\omega^2ab \sin \alpha(2ab \cos 2\alpha - b^2e^{-2\eta_0} - a^2e^{2\eta_0})$$

when $n = 1$,

$$-24\omega Ua^2b \cos 3\alpha$$

when $n = 2$, and

$$108\omega^2a^2b \sin 4\alpha$$

when $n = 3$.

The moment is therefore zero except when $n = 1, 2$ or 3. It is

$$2\pi\varrho U^2ab \sin 2\alpha + 4\pi\varrho\omega^2a^2b(ae^{2\eta_0} - b \cos 2a) \sin 2\alpha$$

when $n = 1$,

$$24\pi\varrho\omega Ua^2b \cos 3\alpha$$

when $n = 2$,

$$-108\pi\varrho\omega^2a^3b \sin 4\alpha$$

when $n = 3$.

In the particular case $n = 1$, $b = a$, the cross-section of the cylinder is an ellipse of semi-axes

$$a' = 2a \cosh \eta_0 \qquad b' = 2a \sinh \eta_0$$

In this case

$$X = \pi\varrho\omega U \sin \alpha[(a'+b')^2 - (a'^2 - b'^2) \cos 2\alpha]$$
$$Y = -\pi\varrho\omega U \cos \alpha[(a'+b')^2 - (a'^2 - b'^2) \cos 2\alpha].$$

and the moment about the centre of the ellipse is

$$\pi\varrho U^2(a'^2 - b'^2) \sin \alpha \cos \alpha + \tfrac{1}{4}\pi\varrho\omega^2(a'^2 - b'^2) \sin 2\alpha[(a'+b')^2 - (a'^2 - b'^2) \cos 2\alpha]$$

These results agree with those given by Roy (1937).

3. We now consider the case when the motion at infinity is given by

$$u = -U \cos \alpha - \omega y, \qquad v = -U \sin \alpha + \omega x$$

The stream function of the undisturbed motion is

$$\begin{aligned}\psi_0 &= U(y \cos \alpha - x \sin \alpha) + \tfrac{1}{2}\omega(x^2 + y^2)\\ &= U(be^{-n\eta} \sin n\xi - nae^{\eta} \sin \xi) \cos \alpha - U(nae^{\eta} \cos \xi + be^{-n\eta} \cos n\xi) \sin \alpha\\ &\qquad + \tfrac{1}{2}\omega[n^2a^2e^{2\eta} + b^2e^{-2n\eta} + 2nabe^{-(n-1)\eta} \cos (n+1)\xi]\end{aligned}$$

The stream function of the disturbed motion is obtained by adding to ψ_0 a term which is the imaginary part of

$$f_2 = Ae^{i\xi} + De^{in\xi} + Ee^{i(n+1)\xi}$$

where A, D, E are given by (5)

The complex velocity is

$$u - iv = M' + N'\frac{d\zeta}{dz}.$$

where

$$M' = -Ue^{-i\alpha} - i\omega(nae^{i\zeta+2\eta_0} + be^{-n\zeta-2n\eta_0})$$

$$N' = -i[Ae^{i\zeta} + nDe^{in\zeta} + (n+1)Ee^{i(n+1)\zeta}],$$

Proceeding, as in the first case, we get

$$X = 2n^2\pi\rho\omega Ua^2e^{2\eta_0}\sin\alpha, \quad Y = -2n^2\pi\rho\omega Ue^{2\eta_0}\cos\alpha$$

and

$$\text{Moment} = 0$$

except when $n = 1$. When $n = 1$,

$$X = 2\pi\rho\omega Ua(ae^{2\eta_0} + b)\sin\alpha$$

$$Y = -2\pi\rho\omega Ua(ae^{2\eta_0} - b)\cos\alpha$$

and

$$\text{Moment} = 2\pi\rho\omega U^2ab\sin 2\alpha$$

In the particular case of the elliptic cylinder of semi-axes a', b'

$$X = \pi\rho\omega Ua'(a' + b')\sin\alpha.$$

$$Y = -\pi\rho\omega Ub'(a' + b')\cos\alpha$$

$$\text{Moment} = \pi\rho U^2(a'^2 - b'^2)\sin\alpha\cos\alpha.$$

These results agree with those given by Roy (1937).

In conclusion, I wish to express my thanks to Dr. S. Ghosh, D.Sc., for suggestions and help.

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CONSIDERATION OF ENERGY FLOW IN THE VIBRATION OF A STELLAR MODEL

BY

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Summary—In this paper we have considered the problem of vibration of stellar model making use of the Energy Equation instead of the condition of adiabacy. It is shown that for opacity law of special kind if there be any stationary vibration of the stellar configuration there must be a node at the surface of the star for the velocity thereby suggesting that there can he no wave of strictly stationary type. The situation does not arise when strict adiabacy is assumed.

Introduction—So far many authors have considered vibration of stellar models making use of the equation of motion, the equation of continuity and a compressibilty equation connecting the pressure and density. But the discussion has so far been limited only to the adiabatic relation $p = k\varrho^{\gamma}$, between p and ϱ such a hypothesis led to the result that there cannot be any phase-lag between the displacement and luminosity pulsation, which is contrary to the observational fact. Under very general circumstances Milne introduced the idea that the luminosity variation should be measured by the rate at which the energy of the configuration changes. But even in such cases as was found by Milne great difficulties arise in finding out the distribution of velocity, displacement etc. inside the star, which are required in setting up the expression for the energy of the stellar configuration . Milne in order to compute the expression for these avoided such difficulties by considering a very symplifying assumption namely,

$$\frac{\delta r}{r} = \frac{\delta R}{R}$$

which is far from being wholly correct.

As a matter of fact the assumption of adiabacy avoids consideration of the equation of energy. We shall in this note replace the condition of adiabacy by the equation for the flow of energy inside the star.

We shall start with the following equations:

(i) the Equation of Motion,
(ii) the Equation of Continuity,
(iii) the Equation of Energy Flow.

The first two equations are well known but we have to set up the equation of energy flow in the stellar interior. It will be shown from these equations that stationary waves of the type usually assumed cannot exist in the configuration we are considering here.

The Equation of the Energy Flow: We make use of the general energy flow equation given by Rosseland [(1949) Eq. (2.7) of Chapter II] in the form

$$\varrho \frac{dE}{dt} + p \operatorname{div} \mathbf{V} = \varrho\varepsilon + \Phi - \operatorname{div} \mathbf{F} \tag{1}$$

In this equation E denotes the internal energy per unit of mass at any point inside the star. This E can be expressed in terms of density and temperature as follows:—

$$E = C_v T + \frac{aT^4}{\varrho} \tag{2}$$

Substitituting this expression of E in equation (1) and remembering that

$$p_{total} = p_{gas} + \tfrac{1}{3}aT^4$$

we get

$$\varrho\frac{d}{dt}(C_v T + 1/\varrho aT^4) + (p + \tfrac{1}{3}aT^4)\,\mathrm{div}\,\mathbf{V} = \varrho\varepsilon + \Phi - \mathrm{div}\,\mathbf{F} \tag{3}$$

As regards $\mathbf{F}$ we express it as

$$F = \frac{C}{\kappa\varrho}pr = \frac{C}{3\kappa\varrho}\frac{\partial}{\partial r}(aT^4) \tag{4}$$

So finally we arrive at the energy equation for the stellar interior as in the form

$$\varrho\frac{d}{dt}(C_v T) + \frac{d}{dt}(aT^4) - (p + \tfrac{4}{3}aT^4)\frac{1}{\varrho}\frac{d\varrho}{dt} = \frac{1}{r^2}\left(r^2\frac{C}{3\kappa\varrho}\frac{\partial}{\partial t}(aT^4)\right) \tag{5}$$

This energy equation has got to be used along with the equation of motion and the equation of continuity for the determination of the motion occuring in the stellar interior. Those two equations are

$$\frac{d}{dt}\mathbf{V} = \mathbf{P} - \frac{1}{\varrho}\,\mathrm{grad}\,p \tag{6}$$

$$\frac{d\varrho}{dt} + \varrho\,\mathrm{div}\,\mathbf{V} = 0 \tag{7}$$

with these we have to consider equation (5).

We use these equations to consider the pulsations of a stellar model.

In order to be able to follow the motion analytically we have to consider a model in which the integration can be effected to a great extent completely. Here an opacity law has been assumed for which the equilibrium configuration is given by simple function. This law is indeed entirely hypothetical but the subsequent integration at least show the behaviour to which the stellar configuration subject to the equation would tend.

The Equilibrium Structure—The equations of equilibrium are, as usual

$$\frac{dp}{dr} = -\frac{GM_r\varrho}{r^2} \tag{8}$$

$$\frac{dM_r}{dr} = 4\pi r^2\varrho \tag{9}$$

and

$$\frac{d}{dr}(\tfrac{1}{3}aT^4) = -\frac{\kappa L}{4\pi cr^2}\,\varrho. \tag{10}$$

Here in the last equation (10) we take an expression for κ different from any existing expression not on any physical grounds but for the fact that for this law of variation of κ the above equations have got simple integrals in terms of known functions in closed form. Such configurations are amenable to analytical treatment. We take an expression for κ as follows

$$\kappa = \kappa_0 \varrho^{-5/2} T^3 \tag{11}$$

Using this expression for κ we easily find the integrals of the above equation to be

$$p = 2\pi G R^4 \varrho_0^2 \left(\frac{1}{r^2} - \frac{1}{R^2}\right) \tag{12}$$

$$M_r = 4\pi \varrho_0 R^2 r \tag{13}$$

$$\varrho = \varrho_0 \frac{R^2}{r^2} \tag{14}$$

These above expressions are the values of these quantities in the equilibrium stage.

The Pulsation Equations:—To study the pulsation of the model the equilibrium configuration of which has just been found out, we proceed with equations (6), (7) and (5). For small vibration we suppose the pulsation amplitude to be a quantity of the first order equations (6), (7) and (5) given

$$\frac{d^2V}{dt^2} = -\frac{1}{\varrho}\frac{\partial}{\partial r}\left(\frac{\partial p_1}{\partial t} + V\frac{\partial p_o}{\partial r}\right) + 4\frac{g_o V}{r} \tag{15}$$

$$\frac{dT_1}{dt} + V\frac{\partial T_o}{\partial r} = \frac{(\gamma-1)T_o}{\varrho}\frac{\partial \varrho_1}{\partial t} + \frac{(\gamma-1)\mu}{a R \varrho_o}\left[\varrho_o^{3/2}\frac{\partial^2 T_1}{\partial r^2} + \frac{\partial T_o}{\partial r}\frac{3}{2}\varrho_o^{\frac{1}{2}}\frac{\partial \varrho_1}{\partial r} + \frac{2}{r}\varrho_o^{3/2}\frac{\partial T_o}{\partial r}\right] \tag{16}$$

$$\frac{\partial \varrho_1}{\partial t} + V\frac{\partial \varrho_o}{\partial r} = -\varrho_o\frac{\partial V}{\partial r} - \frac{2\varrho_o V}{r} \tag{17}$$

In these equations r really denotes r_o and as usual quantities with suffix 0 denote their equilibrium values. The quantities p_1, T_1,...etc. denote the deviations in the values of p, T...etc. from their equlibrium values p_o, T_o,... so that $p = p_o + p_1$, $T = T_o + T_1$ and so on. To solve these equations we follow a particular method. We cannot find the exact solutions of these equations. But since we are interested only in the values of the quantities V, T etc., in the neighbourhood of the boundary we find expressions for these quantities in the neighbourhood of the stellar surface, $r = R$

Let us now therefore put

$$z \equiv R - r$$

and

$$V = (\alpha + \beta z + \gamma z^2 + \ldots\ldots)\, e^{int} \tag{18}$$

$$T_1 = (az + bz^2 + \ldots\ldots)\, e^{int} \tag{19}$$

$$\varrho_1 = (\lambda + \mu z + \nu z^2 + \ldots\ldots)\, e^{int} \tag{20}$$

where α, β, r, a, b, λ, μ, ν are constants which may be real or complex.

We take T_1 to begin with the term z (the absence of the constant term) since T_1/T_0 should be finite at the boundary, *ie.* at $z = 0$, But T_0 is at least of the order of z when z tends to zero. Naturally therefore T_1 must begin with the term. We shall now assume that σ, β, γ etc. in (18) are real. This means we are contemplating some stationary vibration for the velocity inside the star. Let us now substitute the expressions (18), (19) and (20) in the equations (15), (16) and (17), and equate powers of z. Then we get

$$n^2\alpha = 4\pi G\varrho_0(Rp-\alpha)-in\{(R/\mathfrak{u})a+4\pi GR\lambda\} \tag{21}$$

$$n^2\beta = 4\pi G\varrho_0\left(2\gamma R-\frac{2\alpha}{R}\right)in\left\{\frac{4R}{\mathfrak{u}R}a+\frac{2R}{\mu}b+8\pi GR\mu\right\} \tag{22}$$

$$in\lambda+\frac{2\varrho_0}{R}\alpha = \varrho_0\beta-\frac{2\varrho_0}{R}\alpha \tag{23}$$

$$in\mu-\frac{2\varrho_0}{R}\left(\beta-\frac{\alpha}{R}\right) = 2\varrho_0\gamma-\frac{2\varrho_0}{R}\beta-\frac{2\varrho_0}{R}\left(\beta-\frac{\alpha}{R}\right) \tag{24}$$

$$\alpha\left[\frac{2(\gamma-1)\mathfrak{u}}{\mathfrak{a}R\varrho_0R}\frac{\beta}{in}\varrho_0^{5/2}-\beta\varrho_0R\right]=\frac{(\gamma-1)\mathfrak{u}}{\mathfrak{a}R\varrho_0R^2}\left[\frac{3}{2}R^3\varrho_0^{3/2}\beta\mu+2R^2\varrho_0^{3/2}b\right] \tag{25}$$

Here in these equations excepting α, β, γ, the other quantities such as λ, μ, a, b etc. are all supposed to be in general complex. So that if we now put

$$\lambda = \lambda_1+i\lambda_2, \quad \mu = \mu_1+i\mu_2, \quad a = a_1+ia_2 \text{ etc.}$$

we find that the above equations (21-25), when real and imaginary parts are separated reduced to

$$(n^2+4\pi G\varrho_0)\alpha-4\pi G\varrho_0R\beta = \frac{R}{\mathfrak{u}}na_2+4\pi GRn\lambda_2 \tag{26}$$

$$\frac{R}{\mathfrak{u}}a_1+4\pi GR\lambda_1 = 0 \tag{28}$$

$$\frac{8\pi G\varrho_0}{R}\alpha+n^2\beta-3\pi G\varrho_0R\gamma = \frac{4R}{\mathfrak{u}R}na_2+\frac{2Rnb_2}{\mathfrak{u}}+8\pi GRn\mu_2 \tag{27}$$

$$\frac{4}{\mathfrak{u}}\frac{R}{R}a_1+\frac{2R}{\mathfrak{u}}b_1+8\pi GR\mu_1 = 0 \tag{29}$$

$$\lambda_1 = 0 \tag{30}$$

$$n\lambda_2 = \frac{4\varrho_0}{R}\alpha+\varrho_0\beta \tag{31}$$

$$\mu_1 = 0 \tag{32}$$

$$n\mu_2 = \frac{2\varrho_0}{R}\beta-2\varrho_0\gamma \tag{33}$$

$$q\mu_2+sb_2 = -p_1\alpha \tag{34}$$

$$-p_2\alpha = sb_1 \tag{35}$$

where

$$p_1 = (2\gamma-1)\frac{\mathfrak{u}}{\alpha RR}\,\frac{\beta}{n}\,\varrho_o^{3/2} \tag{36}$$

$$p_2 = \beta\varrho_o R \tag{37}$$

$$q = (\gamma-1)\,\frac{\mathfrak{u}}{\alpha R}\,\frac{3}{2}R\varrho_o^{\frac{1}{2}}\beta \tag{38}$$

$$s = (\gamma-1)\,\frac{\mathfrak{u}}{\alpha R}\,R\varrho_o^{\frac{1}{2}} \tag{39}$$

Obviously from equations (27), (30), (32) and (29) we get

$$a_1 = \lambda_1 = \mu_1 = b_1 = 0 \tag{40}$$

Equations (40) lead to the important condition by equation (35) which really means that the stationary vibrations of velocity must have a node on the surface of the star. The other equations like (26), (28) etc. determine the quantities a_2, λ_2 etc. which we shall not discuss here.

Let us now consider the corresponding case of adiabatic pulsation, the equation of energy being dropped. Since we are considering $T_o = 0$ at $z = 0$ and $\varrho_o \neq 0$ at $x = 0$ we have,

$$\frac{T_o}{\gamma\varrho_o} = k_o z + k_1 z^2 + k_2 z^3 + \ldots\ldots \qquad \text{(say)} \tag{41}$$

Again because of the adiabatic hypothesis

$$T_1 = \varrho_1\frac{T_o}{\gamma\varrho_o} \tag{42}$$

Hence using the expressions (19) and (20) we get from equation (42) and (41),

$$\begin{aligned}&(k_o z + k_1 z^2 + \ldots\ldots)(\lambda + \mu z + \ldots\ldots)\\ &= az + bz^2 + \ldots\ldots)\end{aligned} \tag{43}$$

Equating therefore the coefficients of powers of in (43) we have,

$$a_2 = k_o\lambda_2,\ b_2 = k_1\lambda_1 + k_o\mu_2,\ a_1 = k_o\lambda_1 \text{ etc.} \tag{44}$$

where $a = a_1 + ia_2$ etc.

Again with the previous assumption, *viz.*, that α, β, γ's in expression (18) are real, we obtain from equations (27) and (30) which are valid in the adiabatic case also (being the equation of motion and the equation of continuity).

$$\lambda_1 = a_1 = 0 \tag{45}$$

as in equation (40), but they in no way now lead to Hence we conclude that in the adiabatic case the surface is not a node in the case of stationary vibration. Another important conclusion also follows.

Remembering that $\lambda = \lambda_1 + i\lambda_2$ etc. we see that the quantities λ, a, a etc are purely imaginary as $\lambda_1 = a_1 = 0$ etc. Also by hypothesis α is real, so that on the surface of the star *i.e.* at $z = 0$ considering the expression V and T_1/T_0 from equations (18) and (20) we have

$$V = \alpha e^{int} \text{ and } \frac{T_1}{T_0} = \frac{\lambda_2}{\gamma\varrho_0} i . e^{int} = \frac{\lambda_2}{\gamma\varrho_0} e^{i(nt+\pi/2)}.$$

This shows that the temperature and the velocity variations have a difference of phase equal to $\frac{1}{2}\pi$; or in other words, the temperature at the surface of the star is in phase with the displacement. This implies that no phase lag in the usual sense exists.

Conclusion:—In the treatment of the vibration of a stellar configuration the usual condition of adiabacy is replaced by the strict equation of energy. To get an integrable case a hypothetical law of opacity is assumed. It is then shown that for general non-adiabatic vibration of small amplitude a stationary vibration of velocity should have a node at the surface contrary to observational facts. It is however shown that this situation does not arise in the special case of adiabacy when further the temperature fluctuation is in phase with displacement.

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NOTE ON LEGENDRE POLYNOMIALS

BY

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1. T. S. Nanjundiah (1950) gives a proof of Turán's unequality

$$\Delta_n(x) = P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geqq 0 \qquad (-1 \leq x \leq 1,\ n \geq 1) \tag{1.1}$$

that depends on the formula

$$(1-x^2)D_n(x) = n(n+1)\Delta_n(x), \tag{1.2}$$

where

$$D_n(x) = P_n'^2(x) - P_{n-1}'(x)P_{n+1}'(x). \tag{1.3}$$

[See the review by G. Szëgo, *Mathematical Reviews*, vol. 13 (1952), p. 554]. For other proofs of (1.1) see Szëgo (1948). It is easy to prove (1.2) by means of the recursion formulas for $P_n(x)$ and $P_n'(x)$. We wish to point out the (1.2) characterizes the Legendre polynomials More precisely we prove the following

Theorem. *Let $\{f_n(x)\}$ be a set of polynomials, deg $f_n(x) = n$, $f_0(x) = = 1$, $f_1(x) = x$,* satisfying

$$(1-x^2)(f_n'^2 - f_{n-1}'f_{n+1}') = n(n+1)(f_n^2 - f_{n-1}f_{n+1}), \qquad (n \geqq 1). \tag{1.4}$$

Then $f_n(x) = P_n(x)$ for all $n = 0, 2, 3, \ldots\ldots$

Proof. Assume $f_r(x) = P_r(x)$ for $0 \leq r \leq n$. Then (1.4) may be written in the form

$$(1-x^2)(P_n'^2 - P_{n-1}'f_{n+1}') = n(n+1)(P_n^2 - P_{n-1}f_{n+1}). \tag{1.5}$$

We wish to show that (1.5) uniquely determines $f_{n+1}(x)$. Otherwise $f_{n+1}(x) = P_{n+1}(x) + g(x)$, $\deg g(x) \leq n+1$. Examination of the coefficient of x^{2n} in both members of (1.5) shows that the coefficient of x^{n+1} in f_{n+1} is uniquely determined; consequently $\deg g(x) \leq n$. Then (1.5) gives

$$(1-x^2)P_{n-1}'g' = n(n+1)P_{n-1}g. \tag{1.6}$$

Since the polynomials $(1-x^2)P_{n-1}'(x)$ and $P_{n-1}(x)$ have no factor in common, it follows from (1.6) that

$$g(x) = C(1-x^2)P_{n-1}'(x), \qquad g'(x) = Cn(n+1)P_{n-1}(x), \tag{1.7}$$

where C is a constant. If $C \neq 0$, then (1.7) implies

$$(1-x^2)P_{n-1}''(x) - 2xP_{n-1}'(x) = n(n+1)P_{n-1}(x).$$

Comparing this with the familiar differential equation

$$(1-x^2)P_{n-1}''(x) - 2xP_{n-1}'(x) + n(n-1)P_{n-1}(x) = 0$$

we reach a contradiction. This proves the theorem.

Alternatively one can prove by the method of undetermined coefficients that f_{n+1} is is uniquely determined by (1.5).

2. Du Plessis (1951) has given a very simple proof of Grosswald's formula (Grosswald, 1950)

$$\left(\frac{d}{dx}\right)^r P_n(1) = \frac{(n+r)!}{2^r r!(n-r)!} \tag{2.1}$$

using only the generating function af $P_n(x)$. In the same way one can prove

$$\left(\frac{d}{dx}\right)^r P_n^{(\lambda)}(1) = \frac{(2\lambda)_{n+r}}{2^r(\lambda+\frac{1}{2})_r(n-r)!} \tag{2.2}$$

$$\left(\frac{d}{dx}\right)^r P_n^{(\lambda)}(-1) = \frac{(-1)^{n-r}(2\lambda)_{n+r}}{2^r(\lambda+\frac{1}{2})_r(n-r)!},$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ and $P_n^{(\lambda)}(x)$ denotes the ultraspherical polynomial of degree n [for notation see (Szĕgo. 1939), p. 80]. Differentiating both members of

$$(1-2xw+w^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x)w^n$$

r times with respect to x, we get

$$2^r(\lambda)_r w^r(1-2xw+u^2)^{-\lambda-r} = \sum_{n=0}^{\infty}\left(\frac{d}{dx}\right)^r P_n^{(\lambda)}(x)w^n.$$

For $x = 1$, the left member of this equation becomes

$$2^r(\lambda)_r w^r(1-w)^{-2(\lambda+r)} = 2^r(\lambda)_r w^r \sum_{n=0}^{\infty}\frac{(2\lambda+2r)_n}{n!}w^n.$$

Hence

$$\left(\frac{d}{dx}\right)^r P_n^{(\lambda)}(1) = 2^r(\lambda)_r\frac{(2\lambda+2r)_{n-r}}{(n-r)!} = \frac{(2\lambda)_{n+r}}{2^r(\lambda+\frac{1}{2})_r(n-r)!},$$

which proves the first of (2.2). The proof of the second part is exactly the same.

If $P_n^{(\alpha,\beta)}(x)$ denotes the general Jacobi polynomial of degree n, then [Szĕgo, (1939), pp. 61, 62],

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}F\left(-n,\ n+\alpha+\beta+1;\ \alpha+1,\ \frac{1-x}{2}\right) \tag{2.3}$$

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{2.4}$$

Repeated application of (2.4) yields

$$\left(\frac{d}{dx}\right)^r P_n^{(\alpha,\beta)}(x) = \frac{1}{2^r}(n+\alpha+\beta+1)_r P_{n-r}^{(\alpha+r,\beta+r)}(x). \tag{2.5}$$

Since it is clear from (2.3) that

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n},$$

we get using (2.5)

$$\left(\frac{d}{dx}\right)^r P_n^{(\alpha,\beta)}(1) = \frac{1}{2^r}(n+\alpha+\beta+1)_r\binom{n+\alpha}{n-r} = \frac{(n+\alpha+\beta+1)_r\,(n+\alpha)_{n-r}}{2^r(n-r)\,!} \tag{2.6}$$

Similarly the formula [Szëgo, (1939), p. 58]

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$$

yields

$$\left(\frac{d}{dx}\right)^r P_n^{(\alpha,\beta)}(-1) = (-1)^{n-r}\left(\frac{d}{dx}\right)^r P_n^{(\beta,\alpha)}(1),$$

and therefore by (2.6)

$$\left(\frac{d}{dx}\right)^r P_n^{(\alpha,\beta)}(-1) = (-1)^{n-r}\frac{(n+\alpha+\beta+1)_r(n+\beta)_{n-r}}{2^r(n-r)\,!}. \tag{2.7}$$

Since

$$P_n^{(\lambda)}(x) = \frac{(2\alpha+1)_n}{(\alpha+1)_n}P_n^{(\alpha,\alpha)}(x) \qquad (\alpha = \lambda-\tfrac{1}{2}) \quad .$$

we see that (2.6) implies

$$\left(\frac{d}{dx}\right)^r P_n^{(\lambda)}(1) = \frac{1}{2^r}\frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n}\frac{(n+2\lambda)_r(n+\lambda-\frac{1}{2})_{n-r}}{(n-r)\,!} = \frac{(2\lambda)_{n+r}}{2^r(\lambda+\frac{1}{2})_r(n-r)\,!},$$

which is identical with the first of (2.2). In the same way we can verify that (2.7) reduces to the second of (2.2). Finally for $\lambda = \frac{1}{2}$, (2.2) reduces to (2.1).

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ON INTEGRAL REPRESENTATIONS OF MACROBERT'S *E*-FUNCTION

By

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The object of the present paper is to obtain certain integral representations of MacRobert's E-Function and to deduce integral representations of allied functions.

Let
$$\varphi(p) \doteqdot f(t). \quad \text{Then} \quad \varphi(p)/p = \int_0^\infty e^{-pt} f(t)\,dt$$

Let us write $p+\frac{1}{4}a^2/p$ for p. Then since
$$K_\nu(z) = \tfrac{1}{2}(\tfrac{1}{2}z)^\nu \int_0^\infty \exp\{-(t+\tfrac{1}{4}z^2/t)\}t^{-\nu-1}dt,$$
we obtain, on changing the order of integration on the right hand side, where we assume that such a process is justifiable,
$$\int_0^\infty dp[\varphi(p+\tfrac{1}{4}a^2/p)/\{p^{\nu+1}(p+\tfrac{1}{4}a^2/p)\}] = 2^{\nu+1}/a^\nu \int_0^\infty f(t)K_\nu(at)\,dt. \tag{1}$$

Let us put $\nu = -\frac{1}{2}$. We get
$$\int_0^\infty dp[\varphi(p+\tfrac{1}{4}a^2/p)\{p^{\frac{1}{2}}(p+\tfrac{1}{4}a^2/p)\}] = \pi^{\frac{1}{2}} \int_0^\infty t^{-\frac{1}{2}}e^{-at}f(t)\,dt. \tag{A}$$

Let $f(t) = \Gamma(\alpha)t^{\beta-\frac{1}{2}}(1+t)^{-\alpha}$; $\varphi(p) = E(\alpha,\beta+\frac{1}{2}::p)p^{-(\beta-\frac{1}{2})}$

Substituting in (A) and writing x for p, we get
$$\begin{aligned}&\int_0^\infty dx[E(\alpha,\,\beta+\tfrac{1}{2}::x+\tfrac{1}{4}a^2/x)\{x^{\frac{1}{2}}(x+\tfrac{1}{4}a^2/x)\}]\\ &= \pi^{\frac{1}{2}}\Gamma(\alpha)\int_0^\infty e^{-at}t^{\beta-1}(1+t)^{-\alpha}dt\\ &= \pi^{\frac{1}{2}}a^{-\beta}E(\alpha,\,\beta::a),\ R(\alpha)>0,\ R(\beta)>0.\end{aligned}$$

If we write $x = \frac{1}{2}ae^\theta$, then $x+\frac{1}{4}a^2/x = a\cosh\theta$, (1) reduces to
$$\int_{-\infty}^\infty d\theta[E(\alpha,\,\beta+\tfrac{1}{2}::a\cosh\theta)e^{\frac{1}{2}\theta}/\{\cosh^{\beta+\frac{1}{2}}\theta\}] = (2\pi)^{\frac{1}{2}}E(\alpha,\,\beta::a) \tag{2}$$
or
$$\int_0^\infty d\theta[E(\alpha,\,\beta+\tfrac{1}{2}::a\cosh\theta)\cosh\tfrac{1}{2}\theta/\cosh^{\beta+\frac{1}{2}}\theta = (\tfrac{1}{2}\pi)^{\frac{1}{2}}E(\alpha,\,\beta::a). \tag{2$'$}$$

If we put $\cosh\theta = z$, this reduces to
$$\int_1^\infty dz[E(\alpha,\,\beta+\tfrac{1}{2}::pz)/\{(z-1)^{\frac{1}{2}}z^{\beta+\frac{1}{2}}\} = \pi^{\frac{1}{2}}E(\alpha,\,\beta::p). \tag{2$''$}$$

In a previous paper (Bose, 1952), we have proved that

$$p^{-(\nu-1)}E(\alpha, \beta :: p) \doteqdot \{\Gamma(\alpha)\Gamma(\beta)/\Gamma(\nu)\}\,{}_2F_1(\alpha, \beta\,;\,\nu\,;\,-t)t^{\nu-1},$$

so that if we interpret (2′), we get, after we have written p for a and divided both sides by $p^{\nu-1}$,

$$\Gamma(\beta+\tfrac12)\int_0^\infty d\theta[{}_2F_1(a, \beta+\tfrac12\,;\,\nu\,;-x/\cosh\theta)\cosh\tfrac12\theta/\cosh^{\beta+\frac12}\theta = (\tfrac12\pi)^{\frac12}\Gamma(\beta)\,{}_2F_1(\alpha, \beta\,;\,\nu\,;-x) \quad (3)$$

Again let us write $1/p$ for p in (2″).
Since

$$pE(\alpha, \beta :: 1/p) \doteqdot 2t^{\frac12(\alpha+\beta)-1}K_{\beta-\alpha}(2t^{\frac12}),$$

we get on interpretation and after a little simplification

$$\int_1^\infty z^{\frac12(\alpha-\beta-\frac12)}K_{\beta-\alpha+\frac12}\{2(tz)^{\frac12}\}(z-1)^{-\frac12}dz = \pi^{\frac12}t^{-\frac14}K_{\beta-\alpha}(2t^{\frac12}) \qquad (4)$$

If we write $\beta-\alpha = n$ and t^2 for t and z^2 for z, the above reduces to

$$\int_1^\infty \frac{[K_{n+\frac12}(2tz)]}{\{z^{n-\frac12}(z^2-1)^{\frac12}\}} = \tfrac12(\pi/t)^{\frac12}K_n(2t). \qquad (4'')$$

We can write (3) in the from

$$\Gamma(\beta+\tfrac12).\int_0^\infty d\theta[(p/\cosh\theta)\,{}_2F_1(\alpha, \beta+\tfrac12\,;\,\nu\,;-p/\cosh\theta)\cosh\tfrac12\theta/\cosh^{\beta-\frac12}\theta]$$
$$= (\tfrac12\pi)^{\frac12}t^{-\frac12}e^{-\frac12 t}W_{\frac12(\alpha+\beta+1)-\nu,\,\frac12(\alpha-\beta)}\,(t) \qquad (5)$$

Let $\tfrac12(\alpha+\beta+1)-\nu = k$, $\tfrac12(\alpha-\beta) = -m$. Since

$$W_{k,m}(z) = W_{k,-m}(z),$$

we get

$$\int_0^\infty d\theta[\exp(-\tfrac12 t\cosh\theta)W_{k+\frac12,\,m+\frac12}(t\cosh\theta)\cosh\tfrac12\theta/\cosh^{m+\frac34}\theta]$$
$$= (\tfrac12\pi)^{\frac12}t^{-\frac12}e^{-\frac12 t}W_{k,m}(t) \qquad (5\,)$$

Let us again write p^2 for a in (2″) and divide both sides by $p^{2\beta}$. Since

$$(1/p^{2\beta-1})E(\alpha, \beta :: p^2) \doteqdot \pi^{\frac12}\{\Gamma(\alpha)/\Gamma(\beta+\tfrac12)\}(\tfrac12 t)^{2\beta-1}\,{}_1F_1(\alpha\,;\,\beta+\tfrac12\,;-\tfrac14 t^2),\ R(\beta) > 0.$$

Putting $z = x^2$ and interpreting, we have

$$\int_0^\infty (t/x)^{2\beta}\,{}_1F_1(\alpha\,;\,\beta+1\,;-\tfrac14 t^2/x^2)(x^2-1)^{-\frac12}dx$$
$$= \{\Gamma(\beta+1)/\Gamma(\beta+\tfrac12)\}\pi^{\frac12}\int_0^t t^{2\beta-1}\,{}_1F_1(\alpha\,;\,\beta+\tfrac12\,;-\tfrac14 t^2)\,dt$$

or

$$\int_1^\infty x^{-2\beta}(x^2-1)^{-\frac12}\,{}_1F_1(\alpha\,;\,\beta+1\,;-\tfrac14 t^2/x^2)dx$$
$$= \tfrac12\pi^{\frac12}\beta^{-1}\{\Gamma(\beta+1)/\Gamma(\beta+\tfrac12)\}\,{}_2F_2(\alpha, \beta\,;\,\beta+\tfrac12,\ \beta+1\,;-\tfrac14 t^2) \qquad (6)$$

We have proved before (Bose) that

$$E(\alpha, \beta :: p^2) \doteqdot \Gamma(\alpha)\Gamma(\beta)\,{}_2F_2(\alpha,\ \beta\,;\,1,\ \tfrac12\,;-\tfrac14 t^2)$$

Therefore on putting p^2 for p and z^2 for z in $(2'')$, we get

$$\int_1^\infty {}_2F_2(\alpha,\ \beta+\tfrac12\,;\,1,\ \tfrac12\,;-\tfrac14x^2/z^2)z^{-2\beta}(z^2-1)^{-\frac12}dz$$

$$=\tfrac12\pi^{\frac12}\{\Gamma(\beta)/\Gamma(\beta+\tfrac12)\}{}_2F_2(\alpha,\ \beta\,;\,1,\ \tfrac12\,;-\tfrac14x^2),\ R(\beta)>0. \tag{7}$$

Similarly, since

$$1/p\,E(\alpha,\ \beta::p^2)\doteqdot\Gamma(\alpha)\Gamma(\beta)x\,{}_2F_2(\alpha,\ \beta\,;\,1,\ \tfrac32\,;-\tfrac14x^2)$$

$$\int_1^\infty {}_2F_2(\alpha,\ \beta+\tfrac12\,;\,1,\ \tfrac32\,;-\tfrac14x^2/z^2)z^{-2\beta}(z^2-1)^{-\frac12}dz$$

$$=\tfrac12\pi^{\frac12}\{\Gamma(\beta)/\Gamma(\beta+\tfrac12)\}{}_2F_2(\alpha,\ \beta\,;\,1,\ \tfrac32\,;-\tfrac14x^2),\ R(\beta)>0. \tag{7'}$$

We know that

$$\exp\,(-\tfrac12t^2)t^{\nu-1}\doteqdot\Gamma(\nu)p\,\exp\,(\tfrac14p^2)D_{-\nu}(p).$$

Let us put

$$f(t)\,=\,\exp\,(-\tfrac12t^2)t^{\nu-1}/\Gamma(\nu)\,;\ \varphi(p)\,=\,p\,\exp\,(\tfrac14p^2)D_{-\nu}(p),\ R(\nu)>0.$$

We get

$$\int_0^\infty\exp\,(\tfrac14a^2\cosh^2\theta)D_{-\nu-1}(a\cosh\theta)\cosh\tfrac12\theta\,d\theta$$

$$=(\tfrac12\pi/a)^{\frac12}\{\Gamma(\nu+\tfrac12)/\Gamma(\nu+1)\}\exp\,(\tfrac14a^2)D_{-(\nu+\frac12)}(a)\,;\ R(\nu)>-\tfrac12 \tag{8}$$

where we have put $\nu+1$ for ν and $p\,=\,\frac12ae^\theta$.

If we put $\sinh\frac12\theta\,=\,x/\sqrt2$, we get

$$\int_0^\infty\exp\,\{\tfrac14a^2(1+x^2)^2\}D_{-\nu-1}\{a(1+x^2)\}\,dx\,=\,\tfrac12(\pi/a)^{\frac12}\{\Gamma(\nu+\tfrac12)/\Gamma(\nu+1)\}$$

$$\times\exp\,(\tfrac14a^2)D_{-(\nu+\frac12)}(a)\,;\ \ R(\nu)>-\tfrac12$$

Again

$$e^{-\sqrt t}t^{\frac12\nu}\doteqdot(2p)^{-\frac12\nu}\Gamma(\nu+2)e^{(\frac{1}{8p})}D_{-(\nu+2)}\{(2p)^{-\frac12}\},$$

we get after making the same substitution as before, and simplifying

$$\int_0^\infty d\theta\,\exp\,\{1/(8a\cosh\theta)\}\,\{\cosh\tfrac12\theta/(\cosh\theta)^{\frac12\nu+1}\}D_{-(\nu+2)}\{(2a\cosh\theta)^{-\frac12}\}$$

$$=\{\pi^{\frac12}/\nu+1\}e^{(\frac{1}{8a})}D_{-(\nu+1)}\{(2a)^{-\frac12}\} \tag{9}$$

Let $\sinh\frac12\theta\,=\,(\tan\theta_1)/\sqrt2$. The above reduces to (on writing θ for θ_1),

$$\int_0^{\frac12\pi}\exp\,\{\cos^2\theta/(8a)\}\,\cos^\nu\theta\,D_{-(\nu+2)}\{\cos\theta/(2a)^{\frac12}\}d\theta$$

$$=(\tfrac12\pi)^{\frac12}(\nu+1)^{-1}e^{(\frac{1}{8a})}D_{-(\nu+1)}\{(2a)^{-\frac12}\},\ R(\nu)>-1 \tag{10}$$

Again we have

$$\{e^{-at}t^{\frac12(\nu-1)}\}/(1+t)^{\frac12\nu}\doteqdot2^{\frac12\nu}\Gamma(\tfrac12\nu+\tfrac12)pe^{\frac14(p+a)}D_{-\nu}\{2^{\frac12}(p+a)^{\frac12}\}/(p+a)^{\frac12}.$$

Substituting in (A), we get

$$2^{\frac{1}{2}\nu+1}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2})\int_0^\infty \exp\{(2p+a)^2/(8p)\}D_{-\nu}\{(2p+a)(2p)^{-\frac{1}{2}}\}(2p+a)^{-1}\,dp$$

$$=\pi^{\frac{1}{2}}\int_0^\infty dt[e^{-2at}t^{\frac{1}{2}\nu-1}(1+t)^{-\frac{1}{2}\nu}],\ R(\nu)>0$$

$$\int_0^\infty \exp\{(2p+a)^2/(8p)\}(2p+a)^{-1}D_{-\nu}((2p+a)(2p)^{-\frac{1}{2}})dp$$

$$=E(\tfrac{1}{2}\nu,\ \tfrac{1}{2}\nu::2a)/\{4\Gamma(\nu)a^{\frac{1}{2}\nu}\}. \tag{11}$$

Next let

$$f(t)=t^{n-\mu}/(1+t)^{n+\mu};\ \varphi(p)=\Gamma(1+n-\mu)p^{\mu}e^{\frac{1}{2}p}W_{-n\ -\mu+\frac{1}{2}}(p);\ R(1+n-\mu)>0.$$

Therefore

$$\Gamma(1+n-\mu)\int_0^\infty p^{-\frac{1}{2}}(p+\tfrac{1}{4}a^2/p)^{\mu-1}\exp\{\tfrac{1}{2}(p+\tfrac{1}{4}a^2/p)\}W_{-n,-\mu+\frac{1}{2}}(p+\tfrac{1}{4}a^2/p)\,dp$$

$$=\pi^{\frac{1}{2}}\int_0^\infty dt[e^{-at}t^{n-\mu-\frac{1}{2}}/(1+t)^{n+\mu}],\ R(n+\tfrac{1}{2}-\mu)>0.$$

Let $p=\frac{1}{2}ae^{\theta}$. The above reduces to

$$\int_0^\infty \cosh^{\mu-1}\theta\exp(\tfrac{1}{2}a\cosh\theta)W_{-n,-\mu+\frac{1}{2}}(a\cosh\theta)\cosh\tfrac{1}{2}\theta\,d\theta$$

$$=(\tfrac{1}{2}\pi)^{\frac{1}{2}}E(n+\mu,\ n-\mu+\tfrac{1}{2}::a)/\{\Gamma(n+\mu)\Gamma(1+n-\mu)a^{n}\},\ R(n-\mu+\tfrac{1}{2})>0. \tag{12}$$

Next let $\alpha>0$, $\beta>0$ and $\alpha+\beta=1$.

We have proved before that (Bose, *loc. cit.*)

$$E(\alpha,\ \beta::ip)E(\alpha,\ \beta::-ip)\doteqdot\{\Gamma(\alpha)\Gamma(\beta)\}^2{}_2F_1(\alpha,\ \beta;1;-t^2)$$

and

$$E(\alpha,\ \beta::p)\doteqdot\Gamma(\alpha)\Gamma(\beta){}_2F_1(\alpha,\ \beta;1;-t)$$

We have by a known formula (Humbert and Mclachlan, 1941)

$$\Gamma(\alpha)\Gamma(\beta){}_2F_1(\alpha,\ \beta;1;-t^2)\doteqdot p\,\pi^{-\frac{1}{2}}\int_0^\infty \exp(-\tfrac{1}{4}p^2x^2)E(\alpha,\ \beta::x^{-2})\,dx. \tag{13}$$

Therefore

$$(\pi^{\frac{1}{2}}/p)E(\alpha,\ \beta::-ip)E(\alpha,\ \beta::-ip)/\{\Gamma(\alpha)\Gamma(\beta)\}$$

$$=\int_0^\infty \exp(-\tfrac{1}{4}p^2x^2)E(\alpha,\ \beta::x^{-2})\,dx. \tag{13$'$}$$

Let us now write $z=x^2$ and $4p$ for p^2. We obtain

$$(\pi p)^{\frac{1}{2}}E(\alpha,\ \beta::2ip^{\frac{1}{2}})E(\alpha,\ \beta::-2ip^{\frac{1}{2}})$$

$$=\Gamma(\alpha)\Gamma(\beta)p\int_0^\infty z^{-\frac{1}{2}}e^{-pz}E(\alpha,\ \beta::z^{-1})\,dz.$$

Hence we see that

$$(\pi p)^{\frac{1}{2}}E(\alpha,\beta::2ip^{\frac{1}{2}})E(\alpha,\beta::-2ip^{\frac{1}{2}})$$
$$\div\Gamma(\alpha)\Gamma(\beta)t^{-\frac{1}{2}}E(\alpha,\beta::t^{-1})\,; \qquad (B)$$

where we have assumed that $\alpha > 0$, $\beta > 0$ and $\alpha+\beta = 1$.

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STRESS DISTRIBUTION IN THIN AEOLOTROPIC PLATES—1*

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1. **Introduction** :—The fundamental equation satisfied by the stress function for a thin aeolotropic plate and its formal solution have been given by Green (1939), who has also considered the stress distribution in an infinite strip of the plate due to forces, either longitudinal or transverse, acting at any internal point, or on the boundary of the strip ; in a semi-infinite plate due to forces acting along or perpendicular to the boundary and in a wedge due to force at the vertex. In all these cases he has used the method of real variables, given by Filon (1903) and subsequently followed by Howland (1929). In another paper Green (1945) has also used the complex variable method for the solution of the fundamental equation satisfied by the stress function and has applied it to solve the problem of a semi-infinite plate acted by an isolated force at an internal point. In the present paper the stress function has been expressed in terms of a function defined by the Fourier Transform of the stress function. The fundamental equation is then reduced to an ordinary differential equation, the solution of which consistent with the boundary conditions, leads immediately to the determination of stresses. Sneddon (1951) has adopted a similar method for the solution of problems of isotropic plates. The following cases will be considered in details:

(1) a semi-infinite plate with a concentrated force acting at an internal point.

(2) a strip of infinite length but of finite breadth acted on (a) by a concentrated force and (b) by an isolated couple within it.

2. **Fundamental Equation and its Solution.** Let us consider an infinite thin elastic aeolotropic plate of constant thickness. The material of the plate is assumed to have two planes of symmetry at right angles to each other and also at right angles to the plane of the plate. Let these planes be taken to be parrallel to the x and y planes, and the z plane is taken parallel to the plane of the plate. Assuming that the plate is in a state of 'Generalised plane stress', that is, the stresses considered are means with respect to the thickness, the mean stresses are given by

$$\widehat{xx} = \frac{\partial^2 \chi}{\partial y^2}, \quad \widehat{yy} = \frac{\partial^2 \chi}{\partial x^2}, \quad \widehat{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} \tag{1}$$

in cartesian coordinates, where χ is the stress function.

The relations between the mean values of the stresses and the strains in the case considered are

$$e_{xx} = \frac{1}{E_1}\widehat{xx} - \frac{\sigma_1}{E_2}\widehat{yy}, \quad e_{yy} = -\frac{\sigma_2}{E_1}\widehat{xx} + \frac{1}{E_2}\widehat{yy}, \quad e_{xy} = \frac{1}{\mu_1}\widehat{xy} \tag{2}$$

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with the further condition that

$$\frac{\sigma_1}{E_2} = \frac{\sigma_2}{E_1} \tag{8}$$

where E_1 and E_2 are Young's moduli in the x and y directions respectively, σ_1 is the Poisson's ratio of the contraction parallel to the x axis to the extension parallel to the y axis, σ_2 is the Poisson's ratio of the contraction parallel to the y axis to the extension parallel to the x axis, and μ_1 is the modulus of rigidity associated with the directions of x and y.

The compatibility equation of the strains is

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y}. \tag{4}$$

This equation together with (1), (2) and (3) gives (Sokolnikoff, 1949).

$$\frac{\partial^4 \chi}{\partial x^4} + 2k \frac{\partial^4 \chi}{\partial x^2 \partial \eta^4} + \frac{\partial^4 \chi}{\partial \eta^4} = 0 \qquad 5)$$

where

$$k = \tfrac{1}{2}(E_1 E_2)^{\frac{1}{2}}\left(\frac{1}{\mu_1} - \frac{2\sigma_1}{F_2}\right) \tag{6}$$

$$\eta = \gamma y \tag{7}$$

and

$$\gamma = (E_1/E_2)^{\frac{1}{4}}. \tag{8}$$

The solution of the equation (5) consistent with the given boundary conditions gives the stress function χ.

In the present discussion we shall frequently use the Fourier Transform Theorem for the rth derivative of a function, (Sneddon, 1951)

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \frac{d^r f(x)}{dx^r} e^{i\alpha x} dx = (-i\alpha)^r \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} f(x) e^{i\alpha x} dx \tag{9}$$

provided the first $(r-1)$ derivatives of $f(x)$ vanishes as $x \to \pm\infty$. The above relation can be written as

$$F^r = (-i\alpha)^r F \tag{10}$$

where $F \equiv \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} f(x) e^{i\alpha x} dx$ is called the Fourier Transform of the function $f(x)$.

Multiplying the relation (5) by $e^{i\xi x}$ and integrating with respect to x between the limits $-\infty$ and $+\infty$ we get by using the theorem (10)

$$\left(\frac{d^4}{d\eta^4} - 2k\xi^2 \frac{d^2}{d\eta^2} + \xi^4\right) G(\eta, \xi) = 0 \tag{11}$$

where

$$G(\eta, \xi) = \int_{-\infty}^{+\infty} \chi(x, \eta) e^{i\xi x} dx \tag{12}$$

Applying another theorem by Fourier (Sneddon, (1951), namely,

$$\text{"if } F(\alpha) = \frac{1}{(2\pi)^{\frac12}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx, \text{ then, } f(x) = \frac{1}{(2\pi)^{\frac12}}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}d\alpha \tag{13}$$

we get from

$$\chi(x, \eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\eta, \xi)e^{-i\xi x}d\xi. \tag{14}$$

Again, the solution of the linear differential equation (11) is

$$G(\eta, \xi) = A(\xi)\cosh \xi\alpha\eta + B(\xi)\sinh \xi\alpha\eta + C(\xi)\cosh \xi\beta\eta + D(\xi)\sinh \xi\beta\eta \tag{15}$$

where A, B, C and D are independent of η, but are, in general, functions of ξ and $\pm\xi\alpha$, $\pm\xi\beta$ are the roots of the characteristic equation in m,

$$m^4 - 2k\xi^2m^2 + \xi^4 = 0 \tag{16}$$

so that

$$\alpha = \{k+(k^2-1^{\frac12}\}^{\frac12}, \quad \beta = \{k-(k^2-1)^{\frac12}\}^{\frac12} \tag{18}$$

$$\alpha\beta = 1 \tag{17}$$

and, α and β are assumed to be real, that is, $k > 1$. Multiplying the equations in (1) by $e^{i\xi x}$ and integrating with respect to x between limits $-\infty$ and $+\infty$, we have, by using the theorem (10)

$$\int_{-\infty}^{\infty}\widehat{xx}e^{i\xi x}dx = \frac{d^2G}{dy^2}\,; \quad \int_{-\infty}^{+\infty}\widehat{yy}e^{i\xi x}dx = -\xi^2G\,; \quad \int_{-\infty}^{\infty}\widehat{xy}e^{i\xi x}dx = i\xi\frac{dG}{dy}. \tag{19}$$

The stresses can now be expressed in terms of G. For, by the theorem (13) we have from (19)

$$\widehat{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d^2G}{dy^2}e^{i\xi x}d\xi\,; \quad \widehat{yy} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\xi^2Ge^{-i\xi x}d\xi\,; \quad \widehat{xy} = \frac{1}{2\pi}\int_{-\infty}^{\infty}i\xi\frac{dG}{dy}e^{-i\xi x}d\xi \tag{20}$$

When the stresses on the boundary are given, the constants A, B, C and D are determined either by the relations (19) or by the relations (20). Thus, when G is determined the relation (14) gives the stress function χ and hence the problem is solved.

The mean displacements u and v along the x and y axes can also be expressed in terms of G. Multiplying the first of the relations (3) by $e^{i\xi x}$ and and integrating with respect to x between the limits $-\infty$ to $+\infty$ and using the theorem (10) and relations (19) we have

$$\int_{-\infty}^{\infty}ue^{i\xi x}dx = i\left[\frac{1}{E_1\xi}\frac{d^2G}{dy^2} + \frac{\sigma_1}{E_2}\xi G\right]$$

from which, by the theorem (13)

$$u = \frac{i}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{E_1\xi}\frac{d^2G}{dy^2} + \frac{\sigma_1}{E_2}\xi G\right]e^{-i\xi x}d\xi \tag{21}$$

Similarly, the third equation of (3), gives

$$v = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{E_1\xi^2}\frac{d^3G}{dy^3} + \left(\frac{\sigma_1}{E_2} - \frac{1}{\mu_1}\right)\frac{dG}{dy}\right]e^{-i\xi x}d\xi. \tag{22}$$

If the function G is even in ξ, the relations (20) reduces to

$$\widehat{xx} = \frac{1}{\pi}\int_0^\infty \frac{d^2G}{dy^2}\cos\xi x\, d\xi\,;\ \widehat{yy} = -\frac{1}{\pi}\int_0^\infty \xi^2 G\cos\xi x\, d\xi\,;\ \widehat{xy} = \frac{1}{\pi}\int_0^\infty \xi\frac{dG}{dy}\sin\xi x\, d\xi \quad (20.1$$

On the other hand, if G is odd in ξ,

$$\widehat{xx} = -\frac{i}{\pi}\int_0^\infty \frac{d^2G}{dy^2}\sin\xi x\, d\eta,\ \widehat{yy} = \frac{i}{\pi}\int_0^\infty \xi^2 G\sin\xi x\, d\xi,\ \widehat{xy} = \frac{i}{\pi}\int_0^\infty \xi\frac{dG}{dy}\cos\xi x\, d\xi \quad (20.2)$$

3. Isolated force in an Infinite plate. As a solution of (5), if we take the stress function

$$\chi = \frac{F}{2\pi(\alpha^2-\beta^2)}\Big[(\sigma_1\gamma^2+\beta^2)\Big\{\tfrac{1}{2}\alpha\gamma y\log(x^2+\alpha^2\gamma^2y^2)+x\tan^{-1}\frac{\alpha\gamma y}{x}\Big\}$$

$$-(\sigma_1\gamma^2+\alpha^2)\Big\{\tfrac{1}{2}\beta\gamma y\log(x^2+\beta^2\gamma^2y^2)+x\tan^{-1}\frac{\beta\gamma y}{x}\Big\}\Big], \quad (23)$$

the corresponding stresses are using $\alpha\beta = 1$,

$$\left.\begin{aligned}
\widehat{xx} &= \frac{F\gamma^3}{2\pi(\alpha^2-\beta^2)}\left[\frac{(\sigma_1\alpha^2\gamma^2+1)\alpha y}{x^2+\alpha^2\gamma^2y^2}-\frac{(\sigma_1\beta^2\gamma^2+1)\beta y}{x^2+\beta^2\gamma^2y^2}\right]\\
\widehat{yy} &= -\frac{F\gamma}{2\pi(\alpha^2-\beta^2)}\left[\frac{(\sigma_1\alpha^2\gamma^2+1)\beta y}{x^2+\beta^2\gamma^2y^2}-\frac{(\sigma_1\beta^2\gamma^2+1)\alpha y}{x^2+\beta^2\gamma^2y^2}\right]\\
\widehat{xy} &= -\frac{F\gamma}{2\pi(\alpha^2-\beta^2)}\left[\frac{(\sigma_1\alpha^2\gamma^2+1)\beta x}{x^2+\alpha^2\gamma^2y^2}-\frac{(\sigma_1\beta^2\gamma^2+1)\alpha x}{x^2+\beta^2\gamma^2y^2}\right].
\end{aligned}\right\} \quad (24)$$

Also, $\widehat{rx} = \widehat{xx}\cos\theta+\widehat{xy}\sin\theta,\ \widehat{ry} = \widehat{xy}\cos\theta+\widehat{yy}\sin\theta.$ (25)

Considering a circular boundary of radius r, the resultant stresses on the boundary in the x and y directions are

$$\int_0^{2\pi}\widehat{rx}\cdot r d\theta = 0\,;\qquad \int_0^{2\pi}\widehat{ry}\cdot r d\theta = F. \quad (26)$$

Hence the value of χ in (23) represents the stress function for an isolated force F in the positive direction of the y-axis applied at the origin. Equivalent results are given by Green (1939, 1945).

4. Isolated Couple in an Infinite Plate. The relation (23) gives the stress function $\chi(x, y)$ due to a force F acting along the positive direction of y-axis. If an equal and opposite force acts at O' at a distance 'a' from the origin O, the corresponding stress function will be $-\chi(x-a, y)$. Hence, for a couple of moment

$$M = Fa \quad (27)$$

having rotation from y to x sense, the stress function is given by

$$\bar{\chi} = \chi(x, y)-\chi(x-a, y),\ \text{that is,}\ \bar{\chi} = a\frac{\partial\chi(x, y)}{\partial x} \quad (28)$$

when 'a' is very small.

Thus from (23) and (28)

$$\bar{\chi} = \frac{M}{2\pi(\alpha^2-\beta^2)}\left[(\sigma_1\gamma^2+\beta^2)\tan^{-1}\frac{\alpha\gamma y}{x}-(\sigma_1\gamma^2+\alpha^2)\tan^{-1}\frac{\beta\gamma y}{x}\right]. \tag{29}$$

The corresponding stresses are

$$\left.\begin{aligned}
\widehat{xx} &= -\frac{M\gamma^3 xy}{\pi(\alpha^2-\beta^2)}\left[\frac{\alpha(\sigma_1\alpha^2\gamma^2+1)}{(x^2+\alpha^2\gamma^2y^2)^2}-\frac{\beta(\sigma_1\beta^2\gamma^2+1)}{(x^2+\beta^2\gamma^2y^2)^2}\right]\\
\widehat{yy} &= \frac{M\gamma xy}{\pi(\alpha^2-\beta^2)}\left[\frac{\beta(\sigma_1\alpha^2\gamma^2+1)}{(x^2+\alpha^2\gamma^2a^2)^2}-\frac{\alpha(\sigma_1\beta^2\gamma^2+1)}{(x^2+\beta^2\gamma^2y^2)^2}\right]\\
\widehat{xy} &= \frac{M\gamma}{2\pi(\alpha^2-\beta^2)}\left[\frac{\beta(\sigma_1\alpha^2\gamma^2+1)(x^2-\alpha^2\gamma^2y^2)}{(x^2+\alpha^2\gamma^2y^2)^2}-\frac{\alpha(\sigma_1\beta^2\gamma^2y^2)(x^2-\beta^2\gamma^2y^2)}{(x^2+\beta^2\gamma^2y^2)^2}\right]
\end{aligned}\right\} \tag{30}$$

In a subsequent paper we shall apply the above method for the solution of the problems (1) and (2) mentioned in the introduction.

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CERTAIN THEOREMS ON SELF-RECIPROCAL FUNCTIONS

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Section I.

1. In this section we first obtain certain new theorems bearing on self-reciprocal relationships between three pairs of functions.

Let $\varphi_1(p)$ and $f(t)$ be two functions satisfying the integral equation

$$\varphi_1(p) = \int_0^\infty \frac{f(t)\,dt}{(t^2+p^2)^{m+1}}, \qquad m > 0.$$

We then prove

Theorem A. *Let $p^{m+\frac{1}{2}}\,\phi_1(p)$ be R_m; then $x^{-m-\frac{1}{2}}f(x)$ is also R_m, provided $x^{-m-\frac{1}{2}}f(x)$ is continuous and absolutely integrable in* $(0, \infty)$.

We shall first establish the following

Lemma 1. Let

$$\int_0^\infty \frac{xf(x)dx}{(x^2+a)^{m+1}} = \int_0^\infty \frac{xF(x)dx}{(x^2+a)^{m+1}},$$

where $m > 0$, $a > 0$ and $f(x)$ and $F(x)$ or more generally $f(x)-F(x)$ are continuous in $(0, \infty)$, then $f(x) \equiv F(x)$.

Let $G(x) = f(x)-F(x)$. Then

$$\int_0 \frac{x\,G(x)\,dx}{(x^2+a)^{m+1}} = 0.$$

Since this integral is uniformly convergent with respect to a in (a_1, a_2), we can differentiate it with respect to a. Differentiating n times we get

$$\int_0^\infty \frac{x\,G(x)\,dx}{(x^2+a)^{m+n+1}} = 1 \tag{1}$$

Now, let $x^2+a = a/z$;
we then get

$$\int_0^1 G\left[\sqrt{\left\{\frac{a(1-z)}{z}\right\}}\right] z^m z^{n-1} dz = 0, \; (n = 1, 2, 3, \ldots\ldots).$$

Let

$$\varphi(z) = G\left[\sqrt{\left\{\frac{a(1-z)}{z}\right\}}\right]$$

then

$$\varphi(1) = G(0) \text{ and } \varphi(0) = \underset{z\to\infty}{Lt.}\, G(z)$$

so that $\varphi(z)$ is continuous in $0 \leqq z \leqq 1$ and we get

$$\int_0^1 \varphi(z) z^m z^{n-1}\,dz = 0, \quad (n = 1, 2, 3\ldots\ldots)$$

Therefore it follows by lemma of Lerch's Theorem

$$\varphi(z) \equiv 0, \text{ or } G(x) \equiv 0.$$

The result follows by similar arguments if we take $x^r f(x)$, and $x^r F(x)$ instead of $xf(x)$ and $xF(x)$, provided $m \geqq \frac{1}{2}(r-1) \geqq 0$.

This lemma can also be proved by the method of Operational Calculus. We, however, give the method of Operational Calculus in proving the following more general result which we shall require in proving a general theorem later on.

Lemma 2. *Let $G(x)$ be continuous in $(0, X)$ and absolutely integrable in $(0, \infty)$ and*

$$\int_0^\infty \frac{G(x)\,dx}{(x^m + a^m)^n} = 0,$$

where a, m, $n > 0$, then $G(x) \equiv 0$.

Let $x^m = z$ and $a^m = p$. Then we have, on interpretation, when $1/p \subset t$

$$\int_0^\infty e^{-zt} z^{(1/m)-1} G(z^{1/m})\,dz = 0.$$

Therefore

$$\int_0^\infty e^{-zt} G_1'(z)\,dz = 0, \tag{2}$$

where

$$G_1(z) = \int_0^z z^{(1/m)-1} G(z^{1/m})\,dz$$

$$= m \int_0^{z^{1/m}} G(u)\,du.$$

Evidently $G_1(z)$ is a continuous function of z and $G_1(0) = 0$. Integrating by parts we get from (2)

$$\int_0^\infty e^{-zt} G_1(z)\,dz = 0.$$

Hence, by Lerch's Theorem, we get $G_1(z) \equiv 0$.

By differentiation it then follows that $G(z) \equiv 0$.

Now we proceed to prove Theorem *A*.

By hypothesis we have

$$p^{m+\frac{1}{2}} \varphi_1(p) = \int_0^\infty \sqrt{(px)}\, J_m(px) x^{m+\frac{1}{2}} \varphi_1(x)\,dx$$

$$= \int_0^\infty \sqrt{(px)}\, J_m(px) x^{m+\frac{1}{2}}\,dx \int_0^\infty \frac{f(y)\,dy}{(x^2+y^2)^{m+1}}$$

Changing the order of integration, we get

$$p^{m+\frac{1}{2}}\varphi(p) = p^{\frac{1}{2}}\int_0^\infty f(y)\,dy\int_0^\infty \frac{x^{m+1}J_m(px)\,dx}{(x^2+y^2)^{m+1}},$$

Hence

$$2^m\Gamma(m+1)\varphi_1(p) = \int_0^\infty f(y)\,K_o(py)\,dy. \tag{3}$$

Again

$$\int_0^\infty f(y)K_o(py)\,dy = 2^m\Gamma(m+1)\int_0^\infty f(y)\,dy\int_0^\infty \frac{y^{-m}x^{m+1}J_m(xy)dx}{(x^2+p^2)^{m+1}}$$

and on changing the order of integration we get

$$\int_0^\infty f(y)K_o(py)\,dy = 2^m\Gamma(m+1)\int_0^\infty \frac{x^{m+1}dx}{(x^2+p^2)^{m+1}}\int_0^\infty y^{-m}J_m(xy)f(y)\,dy, \tag{4}$$

Therefore

$$\varphi_1(p) = \int_0^\infty \frac{x^{m+\frac{1}{2}}F(x)dx}{(x^2+p^2)^{m+1}},$$

where

$$F(x) = \int_0^\infty \surd(xy)J_m(xy)\cdot y^{-m-\frac{1}{2}}f(y)dy,$$

But by our definition

$$\varphi_1(p) = \int_0^\infty \frac{f(x)\,dx}{(x^2+p^2)^{m+1}}.$$

Therefore by the Lemma 1, we get

$$F(x) \equiv x^{-m-\frac{1}{2}}f(x)$$

i.e.

$$\int_0^\infty \surd(xy)J_m(xy)\cdot y^{-m-\frac{1}{2}}f(y)\,dy = x^{-m-\frac{1}{2}}f(x),$$

provided $x^{-m-\frac{1}{2}}f(x)$ is absolutely integrable and continuous in $(0, \infty)$ and $m \geqq \frac{1}{2}$ and $F(\infty)$ exists.

We must now justify the changes in orders of integrations. The changes in the orders of integrations will be justified if the following conditions hold good:

(*a*) $F(x,y)$ is a continuous function of both the variables in $x \geqq 0$, $y \geqq 0$

(*b*) $\int_0^\infty F(x, y)\,dy$ is uniformly convergent in the arbitrary interval $0 \leqq x \leqq a$

(*c*) $\int_0^\infty F(x, y)\,dx$ is uniformly convergent in the arbitrary interval $0 \leqq y \leqq b$

(*d*) $\int_0^\infty dy\int_0^x F(x, y)\,dx$ is uniformly convergent in $x \geqq 0$.

We take

$$F(x, y) = \frac{x^{m+1}J_m(px)f(y)}{(x^2+y^2)^{m+1}}, \quad (R(m) > \tfrac{1}{2})$$

and

$$F_1(x, y) = \frac{x^{-m}y^{m+1}f(x)J_m(xy)}{(y^2+p^2)^{m+1}}$$

The condition $R(m) > \frac{1}{2}$ can now be waived by analytic continuation and the theorem holds when $m \geqq \frac{1}{2}$.

Theorem B. Conversely, let $x^{-m-\frac{1}{2}}f(x)$ be R_m; then $x^{m+\frac{1}{2}}\varphi_1(x)$ is also R_m, provided the integral for the self-reciprocal property exists.

We have

$$x^{-m-\frac{1}{2}}f(x) = \int_0^\infty \sqrt{(xy)}J_m(xy)y^{-m-\frac{1}{2}}f(y)dy$$

The proceeding as before and changing the orders of integrations, we have, as in (3) and (4)

$$\int_0^\infty \sqrt{(px)}J_m(px)x^{m+\frac{1}{2}}\varphi_1(x)dx = \frac{p^{m+\frac{1}{2}}}{2^m\Gamma(m+1)}\int_0^\infty f(y)K_0(py)dy$$

$$= p^{m+\frac{1}{2}}\int_0^\infty \frac{x^{m+\frac{1}{2}}dx}{(x^2+p^2)^{m+1}}\int_0^\infty \sqrt{(xy)}J_m(xy)y^{-m-\frac{1}{2}}f(y)dy$$

$$= p^{m+\frac{1}{2}}\int_0^\infty \frac{f(x)dx}{(x^2+p^2)^{m+1}} = p^{m+\frac{1}{2}}\varphi_1(p),$$

thus showing that $x^{m+\frac{1}{2}}\varphi_1(x)$ is R_m.

The changes in the orders of integrations are justified here as before and assumed permissible.

Let us next consider the pair of functions $f(t)$ and $\psi_1(p)$ which satisfy the integral equation

$$\psi_1(p) = \int_0^\infty \sqrt{(pt)}K_m(pt)f(t)dt$$

When $m = \pm\frac{1}{2}$, this reduces to the Laplace transform

$$\psi_1(p) = \sqrt{\frac{\pi}{2}}\int_0^\infty e^{-pt}f(t)dt,.$$

Theorem C. *Let* $p^{2m}\psi_1(p)$ *be* R_{3m}; *then* $x^{-2m}f(x)$ *is* R_m, *provided* $x^{-2m}f(x)$ *is continuous and absolutely integrable in* $(0, \infty)$ and $m \geqq \frac{1}{2}$.

We have

$$y^{2m}\psi_1(y) = \int_0^\infty \sqrt{(py)}J_{3m}(py).p^{2m}\psi_1(p)dp$$

$$= \int_0^\infty \sqrt{(py)}J_{3m}(py)p^{2m}dp\int_0^\infty \sqrt{(px)}K_m(px)f(x)\,dx.$$

On changing the order of integration, we get

$$y^{2m}\psi_1(y) = \int_0^\infty \sqrt{(xy)}f(x)\,dx \int_0^\infty p^{2m+1}K_{-m}(px)f(x)\,dx$$

$$= \Gamma(2m+1)2^{2m}y^{3m+\frac{1}{2}}\int_0^\infty \frac{x^{\frac{1}{2}-m}f(x)dx}{(x^2+y^2)^{2m+1}}.$$

Therefore

$$\psi_1(y) = 2^{2m}\Gamma(2m+1)\,y^{m+\frac{1}{2}}\int_0^\infty \frac{x^{\frac{1}{2}-m}f(x)dx}{(x^2+y^2)^{2m+1}}.$$

Again, we can write

$$\psi_1(p) = \int_0^\infty \sqrt{(px)}K_m(px)f(x)dx$$

$$= 2^{2m}\Gamma(2m+1)p^{m+\frac{1}{2}}\int_0^\infty x^{\frac{1}{2}-2m}f(x)dx\int_0^\infty \frac{y^{m+1}J_m(xy)\,dy}{(p^2+y^2)^{2m+1}}.$$

On changing the order of integration, we get

$$\psi_1(p) = 2^{2m}\Gamma(2m+1)p^{m+\frac{1}{2}}\int_0^\infty \frac{y^{m+\frac{1}{2}}F(y)\,dy}{(p^2+y^2)^{2m+1}}, \tag{5}$$

where

$$F(y) = \int_0^\infty \sqrt{(xy)}J_m(xy)\,x^{-2m}f(x)\,dx.$$

Hence we get

$$\int_0^\infty \frac{y^{m+\frac{1}{2}}F(y)}{(p^2+y^2)^{2m+1}} = \int_0^\infty \frac{y^{\frac{1}{2}-m}f(y)dy}{(p^2+y^2)^{2m+1}}$$

By Lemma 1, we therefore get

$$F(y) \equiv y^{-2m}f(y),$$

provided $x^{-2m}f(x)$ is continuous and absolutely integrable in $(0, \infty)$ and $m \geqq \frac{1}{2}$ and $F(\infty)$ exists.

In this case we take

$$F(x,\,y) = x^{\frac{1}{2}}y^{2m+1}J_{3m}(py)K_m(xy)f(x)$$

$$F_1(x,\,y) = \frac{x^{\frac{1}{2}-2m}y^{m+1}J_m(xy)f(x)}{(p^2+y^2)^{2m+1}},$$

the ranges of integrations being from ϵ to infinity, ultimately $\epsilon \to 0$.

Theorem D. *Conversely, let* $x^{-2m}f(x)$ *be* R_m; *then* $p^{2m}\,\psi_1(p)$ *is* R_{3m}, *provided the integral for the self-reciprocal property and* $m \geqq \frac{1}{2}$

We have

$$x^{-2m}f(x) = \int_0^\infty \sqrt{(xy)}J_m(xy)\cdot y^{-2m}f(y)\,dy.$$

Then proceeding as in the previous theorem and changing the orders of integrations, we get as in (5)

$$\int_0^\infty \sqrt{(py)}J_{3m}(py)\,p^{2m}\psi_1(p)\,dp$$

$$= \Gamma(2m+1)2^{2m}y^{3m+\frac{1}{2}}\int_0^\infty \frac{x^{\frac{1}{2}-m}f(x)\,dx}{(x^2+y^2)^{2m+1}}$$

and
$$\psi_1(y) = 2^{2m}\Gamma(2m+1)y^{m+\frac{1}{2}}\int_0^\infty \frac{x^{\frac{1}{2}+m}F(x)\,dx}{(x^2+y^2)^{2m+1}}$$
$$= 2^{2m}\Gamma(2m+1)\,y^{m+\frac{1}{2}}\int_0^\infty \frac{x^{\frac{1}{2}-m}f(x)\,dx}{(x^2+y^2)^{2m+1}}.$$

Hence
$$\int_0^\infty \surd(py)J_{3m}(py)\,p^{2m}\psi_1(p)\,dp = y^{2m}\psi_1(y).$$

The changes in the orders of integrations are assumed justified here as in the previous case.

Let us next consider the functions $f(t)$ and $\psi_2(p)$ as well as the pair $f(t)$ and $\psi_3(p)$. where
$$\psi_2(p) = \int_0^\infty (pt)^m K_m(pt) f(t)\,dt \tag{5}$$
and
$$\psi_3(p) = \int_0^\infty (pt)^{m+1}K_m(pt)f(t)\,dt. \tag{5'}$$

When $m = \frac{1}{2}$, (5′) reduces to the Laplace Transform.

Theorem E. *Let $p^{m+\frac{1}{2}}\psi_2(p)$ be R_{3m}; then $x^{-m-\frac{1}{2}}f(x)$ is R_m, provided $x^{-m-\frac{1}{2}}f(x)$ is continuous and absolutely integrable in $(0, \infty)$ and $m \geqq \frac{1}{2}$.*

We have
$$y^{m+\frac{1}{2}}\psi_2(y) = \int_0^\infty \surd(py)J_{3m}(py)\,p^{m+\frac{1}{2}}\psi_2(p)\,dp$$
$$= \int_0^\infty (py)J_{3m}(py)p^{m+\frac{1}{2}}\,dp \int_0^\infty (px)^m K_m(px)f(x)\,dx.$$

On changing the order of integration we get
$$y^{m+\frac{1}{2}}\psi_2(y) = y^{\frac{1}{2}}\int_0^\infty x^m f(x)\,dx\int_0^\infty p^{2m+1}J_{3m}(py)K_{-m}(px)\,dp$$
$$= y^{3m+\frac{1}{2}}2^{2m}\Gamma(2m+1)\int_0^\infty \frac{f(x)\,dx}{(x^2+y^2)^{2m+1}}. \tag{6}$$

Therefore
$$\psi_2(y) = 2^{2m}\Gamma(2m+1)y^{2m}\int_0^\infty \frac{f(x)\,dx}{(x^2+y^2)^{2m+1}}.$$

Again
$$\psi_2(p) = \int_0^\infty (px)^m f(x)K_{-m}(px)\,dx$$
$$= 2^{2m}\Gamma(2m+1)\,p^{2m}\int_0^\infty x^{m}f(x)\,dx\frac{y^{m+1}J_m(xy)\,dy}{(p^2+y^2)^{2m+1}}.$$

On changing the order of integration, we have
$$\psi_2(p) = 2^{2m}\Gamma(2m+1)\,p^{2m}\int_0^\infty \frac{y^{m+\frac{1}{2}}F(y)\,dy}{(p^2+y^2)^{2m+1}}, \tag{7}$$

where
$$F(y) = \int_0^\infty \sqrt{(xy)}J_m(xy)\, x^{-m-\frac{1}{2}}f(x)dx.$$

Hence by Lemma 1, we get
$$F(y) \equiv y^{-m-\frac{1}{2}}f(y)$$

i.e.,
$$\int_0^\infty \sqrt{(xy)}J_m(xy)\, x^{-m-\frac{1}{2}}f(x)dx = y^{-m-\frac{1}{2}}f(y),$$

provided $x^{-m-\frac{1}{2}}f(x)$ is continuous and absolutely integrable in $(0, \infty)$ and $m \geqq \frac{1}{2}$ and $F(\infty)$ exists

Theorem F. *Conversely, let* $x^{-m-\frac{1}{2}}f(x)$ *be* R_m; *then* $p^{m+\frac{1}{2}}\psi_1(p)$ *is* R_{3m}, *provided the integral for the self-reciprocal property exists and* $m \geqq \frac{1}{2}$.

This result can be easily proved by proceeding as before.

Theorem G. *Let* $p^{m-\frac{1}{2}}\psi_3(p)$ *be* R_{8m}; *then* $x^{\frac{1}{2}-m}f(x)$ *is* R_m, *provided* $x^{\frac{1}{2}-m}f(x)$ *is continuous and absolutely integrable in* $(0, \infty)$ *and* $m \geqq \frac{1}{2}$.

If we write in (5′) $xf(x)$ for $f(x)$ and $\psi_3(p)/p$ for $\psi_2(p)$, we get the integral equation for $\psi_3(p)$. Thus Theorem G follows immediately from Theorem E, if we write $xf(x)$ for $f(x)$ and $\psi_3(p)/p$ for $\psi_2(p)$.

In the same way from Theorem F, we reduce

Theorem H. *Conversely, let* $x^{\frac{1}{2}-m}f(x)$ *be* R_m; *then* $p^{m-\frac{1}{2}}\psi_3(p)$ *is* R_{8m}, *provided the self-reciprocal property exists and* $m \geqq \frac{1}{2}$.

2. The following results are immediately deducible from some of the above theorems.

$$\frac{d}{d(pz)}\left\{(pz)^{m+1}K_{m+1}(pz)\right\} = -(pz)^{m+1}K_m(pz),$$

$$\psi_3(p) = -\int_0^\infty \frac{d}{d(pz)}\left\{(pz)^{m+1}K_{m+1}(pz)\right\}f(z)\,dz.$$

On integrating by parts and assuming that $f(0) = 0$ and $f(\infty)$ is finite, we get

$$\psi_3(p) = p^m \int_0^\infty z^{m+1}K_{m+1}(pz)f'(z)\,dz,$$

Therefore
$$p\,\psi_3(p) = \int_0^\infty (pz)^{m+1}K_{m+1}(pz)f'(z)\,dz.$$

Applying Theorem E, we get

Corollary 1 *If* $p^{m+5/2}\psi_3(p)$ *be* R_{8m+8}; *then* $x^{-m-3/2}f'(x)$ *is* K_{m+1} *and conversely.*

Again, let $f'(0)$, $f'(\infty)$ be finite. Then assuming that differentiation under the sign of integration is permissible, we have from

$$\frac{d}{dp}\Big\{p\psi_3(p)\Big\}=\int_0^\infty \frac{d}{dp}\Big\{(pz)^{m+1}K_{m+1}(pz)\Big\}f'(z)\;dz=\frac{1}{p}\int_0^\infty\frac{d}{dz}\Big\{(pz)^{m+1}K_{m+1}(pz)\Big\}zf'(z)\,dz$$

$$p\frac{d}{dz}\Big\{p\psi_3(p)\Big\}=-\int_0^\infty (pz)^{m+1}K_{m+1}(pz)\frac{d}{dz}\Big\{zf'(z)\Big\}dz.$$

Proceeding exactly as before, we get

Corollary 2. *If* $p^{m+5/2}\frac{d}{dp}\Big\{p\psi_3(p)\Big\}$ *is* R_{3m+3}, *then* $x^{-m-3/2}\frac{d}{dx}\Big\{xf'(x)\Big\}$ *is* R_{m+1} *and, conversely.*

Let again $\underset{z\to 0}{Lt.}\frac{d}{dz}[zf'(z)]=0$ and $\underset{z\to\infty}{Lt.}\frac{d}{dz}\;[zf'(z)]$ be finite. Then we get as before

Corollary 3. *If* $p^{m+5/2}\frac{d}{dp}\Big[p\frac{d}{dp}\Big\{p\psi_3(p)\Big\}\Big]$ *is* R_{3m+3}, *then* $x^{-m-3/2}\frac{d}{dx}\Big[x\frac{d}{dx}\Big\{xf'(x)\Big\}\Big]$ *is* R_{m+1} *and conversely.*

Examples:

(1) Let us take $\qquad f(t)=t^{2m+1}\;e^{-\frac{1}{2}t^2}$

then $\qquad t^{-m-\frac{1}{2}}f(t)=t^{m+\frac{1}{2}}\;e^{-\frac{1}{2}t^2},$

which is R_m.

We have

$$\varphi_1(p)=\int_0^\infty e^{-\frac{1}{2}t^2}t^{2m+1}(t^2+p^2)^{-(m+1)}\,dt=p^{-2(m+1)}\int_0^\infty e^{-\frac{1}{2}t^2}t^{2m+1}(1+t^2/p^2)^{-(m+1)}\,dt$$

On putting $\frac{1}{2}t^2=y$, we get

$$\varphi_1(p)=2^m\,p^{-(m+1)}\int_0^\infty e^{-y}y^m\{1+y/(\tfrac{1}{2}p^2)\}^{-(m+1)}\;dy$$

The E-function of Mac-Robert (1942) is defined as

$$E(\alpha,\beta::x)=\Gamma(\alpha)\int_0^\infty e^{-y}y^{\beta-1}(1+y/x)^{-\alpha}dy,\quad R(\beta)>0$$

We, therefore, get

$$\varphi_1(p)=\frac{2^m p^{-2(m+1)}}{\Gamma(m+1)}E(m+1,\;m+1::\tfrac{1}{2}p^2)$$

so that

$$p^{m+\frac{1}{2}}\varphi_1(p)=\frac{2^m p^{-(m+3/2)}}{\Gamma(m+1)}E(m+1,\;m+1::\tfrac{1}{2}p^2)$$

N. N. Bose (1950) has shown that

$$p^{-(\alpha+\frac{1}{2})}E\{\alpha,\;\tfrac{1}{2}(\alpha+m+1)::\tfrac{1}{2}p^2\}$$

is R_m.

Hence taking $\alpha = m+1$, we find that

$$p^{m+\frac{1}{2}}\varphi_1(p)$$

is R_m.

We thus get an example of Theorem B.

(2) We have proved (Bose, 1952)

$$\int_0^\infty \frac{{}_1F_1(\nu\,;1+\varrho\,;\,-\frac{1}{2}x^2)(\frac{1}{2}x^2)^\varrho x\,dx}{(x^2+K^2)^{\varrho+3/2}} = \frac{\Gamma(\nu+\frac{1}{2})\Gamma(\varrho+1)}{\Gamma(\varrho+\frac{3}{2})}\,2^{\nu-1-\varrho}K^{-1}e^{\frac{1}{4}K^2}D_{-2\nu}(K)$$

On writing $\varrho+\frac{1}{2} = m$, $\nu = m$ and $k = p$ we get

$$\int_0^\infty \frac{x^{2m}{}_1F_1(m\,;\,m+\frac{1}{2}\,;\,-\frac{1}{2}x^2)\,dx}{(x^2+p^2)^{m+1}} = \frac{\{\Gamma(m+\frac{1}{2})\}^2}{\Gamma(m+1)}2^{m-1}p^{-1}e^{\frac{1}{4}p^2}D_{-m}(p).$$

Let us take

$$\varphi_1(p) = A\cdot p^{-1}e^{\frac{1}{4}p^2}D_{-2m}(p),\quad \text{where } A = \text{a numerical quantity};$$

then $f(t) = {}_1F_1(m\,;\,m+\frac{1}{2}\,;\,-\frac{1}{2}t^2)t^{2m}$.

Then
$$p^{m+\frac{1}{2}}\varphi_1(p) = Ap^{m-\frac{1}{2}}e^{\frac{1}{4}p^2}D_{-2m}(p)$$
and
$$t^{-m-\frac{1}{2}}f(t) = t^{m-\frac{1}{2}}{}_1F_1(m\,;\,m+\frac{1}{2}\,;\,-\frac{1}{2}t^2),$$

It has been proved by Varma (1937) that

$$p^{m-\frac{1}{2}}e^{\frac{1}{4}p^2}D_{-2m}(p)$$

is R_m. Also B. Mohan (1941) has proved that

$$t^{m-\frac{1}{2}}{}_1F_1(m\,;\,m+\frac{1}{2}\,;\,-\frac{1}{2}t^2)$$

is R_m.

Thus $p^{m+\frac{1}{2}}\varphi_1(p)$ is R_m and so is also $t^{-m-\frac{1}{2}}f(t)$.

The above is therefore an example of Theorem B.

3. We now proceed to a general theorem.

Theorem I. *Let $g(x, p)$ and $h(x, p)$ be two continuous and symmetric functions of x and p in $0 \leqq x \leqq \infty$ and $0 \leqq p \leqq \infty$, such that*

$$\int_0^\infty \sqrt{(xy)}J_m(xy)h(x, p)\,dx = g(y, p),\quad m \geqq 0.$$

Further let $f(t)$ be continuous and absolutely integrable in $(0, \infty)$ and let

$$\int_0^\infty |g(p, t)|\,dt \text{ and } \int_0^\infty |h(p, t)|\,dt$$

exist and be integrable in $(0, \infty)$.

Also let

$$\psi(p) = \int_0^\infty g(x, p)\,f(x)\,dx.$$

Then, if $f(x)$ be R_m, $\psi(p)$ is also R_m.

We have
$$\int_0^\infty \sqrt{(xy)}J_m(xy)\,f(y)\,dy = f(x).$$

Also by Hankel's Theorem

$$\int_0^\infty \sqrt{(ax)}\, J_m(ax)\, g(x, p)\, dx = h(a, p).$$

Then

$$\psi(p) = \int_0^\infty g(x, p)\, dx \int_0^\infty \sqrt{(xy)}\, J_m(xy)\, f(y)\, dy$$

On changing the order of integration, which is obviously permissible, we get

$$\psi(p) = \int_0^\infty f(y)\, dy \int_0^\infty \sqrt{(xy)}\, J_m(xy) g(x,p)\, dx.$$

$$= \int_0^\infty h(y, p)\, f(y)\, dy.$$

Therefore

$$\int_0^\infty \sqrt{(bp)}\, J_m(bp)\, \psi(p)\, dp$$

$$= \int_0^\infty \sqrt{(bp)}\, J_m(bp)\, dp \int_0^\infty h(y, p)\, f(y)\, dy$$

$$= \int_0^\infty f(y)\, dy \int_0^\infty (bp)\, J_m(bp)\, h(y, p)\, dp,$$

on changing the order of integration,

$$= \psi(b)$$

Theorem J. *Conversely, let* $\psi(p)$ *be* R_m. *Then*

$$\psi(b) = \int_0^\infty \sqrt{(pb)}\, J_m(pb)\, \psi(p)\, dp$$

$$= \int_0^\infty \sqrt{(pb)}\, J_m(pb)\, dp \int_0^\infty f(y)\, g(y, p)\, dy$$

$$= \int_0^\infty f(y)\, dy \int_0^\infty \sqrt{(pb)}\, J_m(pb)\, g(y, p)\, dp$$

$$= \int_0^\infty h(y, b)\, f(y)\, dy.$$

By by hypothesis

$$\psi(b) = \int_0^\infty f(x)\, g(x, b)\, dx$$

$$= \int_0^\infty f(x)\, dx \int_0^\infty \sqrt{(xy)}\, J_m(xy)\, h(y, b)\, dy$$

$$= \int_0^\infty h(y, b)\, dy \int_0^\infty \sqrt{(xy)}\, J_m(xy)\, f(x)\, dx$$

$$= \int_0^\infty h(y, b)\, F(y)\, dy,$$

where

$$F(y) = \int_0^\infty \sqrt{(xy)}\, J_m(xy)\, f(x)\, dx.$$

Hence

$$\int_0^x h(y, b)\{F(y)-f(y)\}\, dy = 0 \tag{8}$$

The question now arises when is $F(y) \equiv f(y)$?

Now let $G(y) = F(y)-f(y)$ in (8).

If $h(y, b) = e^{-by}$, then by Lerch's Theorem $G(y) \equiv 0$

Again if $G(y)$ be continuous and absolutely integrable in $(0, \infty)$ and $h(y, b) = 1/(y^m+b^m)^n$, where $b, m, n > 0$, then by Lemma 2, we get $G(y) \equiv 0$.

Section II.

Self-Reciprocal Function involving two variables.

Some of the theorems established before (Mitra and Bose, 1953 and Bose, 1952) and in section I of this paper can be easily extended to two or more variables. We shall content ourselves by proving below only a few theorems extended to two variables. The methods of proofs given for two variables clearly indicate that they can be extended to more than two variables.

1. We first proceed to prove the following :

Lemma 3. Let $F(x_1, x_2)$ be a continuous function of both variables in $0 \leqq x_1 \leqq \infty$, $0 \leqq x_2 \leqq \infty$ and be such that

$$\int_0^\infty \int_0^\infty e^{-(p_1x_1+p_2x_2)} F(x_1, x_2)\, dx_1\, dx_2 = 0,$$

where $p_1, p_2 > 0$; then $F(x_1, x_2) \equiv 0$.

Let

$$\int_0^\infty e^{-p_1x_1} F(x_1, x_2)\, dx_1 = \psi(x_2).$$

Then since obviously the integral converges uniformly in $0 \leqq x_2 \leqq \infty$, $\psi(x_2)$ is a continuous function of x_2 in $(0, X)$. We have, therefore,

$$\int_0^\infty e^{-p_2x_2} \psi(x_2)\, dx_2 = 0$$

where $\psi(x_2)$ is continuous.

Then by Lerch's Theorem $\psi(x_2) \equiv 0$.

Therefore

$$\int_0^\infty e^{-p_1x_1} F(x_1, x_2)\, dx_1 = 0.$$

Hence, by Lerch's Theorem, we finally get $F(x_1, x_2) \equiv 0$.

With the help of Lemma 3 we shall now establish the following :

Theorem K. *Let*

$$\varphi(p_1, p_2) = p_1p_2 \int_0^\infty \int_0^\infty e^{-(p_1x_1+p_2x_2)} f(x_1, x_2)\, dx_1\, dx_2$$

and further let $p_1^{m-\frac{1}{2}}p_2^{n-\frac{1}{2}}\varphi(p_1, p_2)$ *be self-reciprocal in the Hankel Transform of order* $m+1$ *with respect to* p_1 *and of order* $n+1$ *with respect to* p_2. *Then* $x_1^{-m-\frac{1}{2}}x_2^{-n-\frac{1}{2}}f(x_1, x_2)$ *is self-reciprocal in the Hankel Transform of orders* m *and* n *with respect to* p_1 *and* p_2 *respectively, provided* $f(x_1, x_2)$ *is continuous in both the variables in* $0 \leqq x_1 \leqq \infty$ *and* $0 \leqq x_2 \leqq \infty$ *and so is also* $x_1^{m+\frac{1}{2}}x_2^{n+\frac{1}{2}}F(x_1, x_2)$ *which has been defined later on in* (9).

By hypothesis, we have

$$\int_0^\infty \int_0^\infty \sqrt{(p_1x_1)}\sqrt{(p_2x_2)}J_{m+1}(p_1x_1)J_{n+1}(p_2x_2) \cdot \varphi(p_1, p_2)p_1^{m-\frac{1}{2}}p_2^{n-\frac{1}{2}}\, dp_1\, dp_2$$

$$= x_1^{m-\frac{1}{2}}x_2^{n-\frac{1}{2}}\varphi(x_1, x_2)$$

$$= x_1^{m+\frac{1}{2}}x_2^{n+\frac{1}{2}} \int_0^\infty \int_0^\infty e^{-(y_1x_1+y_2x_2)}f(y_1,y_2)\, dy_1\, dy_2$$

We therefore have

$$x_1^{m}x_2^{n} \int_0^\infty \int_0^\infty e^{-(y_1x_1+y_2x_2)}f(y_1, y_2)\, dy_1\, dy_2$$

$$= \int_0^\infty \int_0^\infty p_1^{m+1}p_2^{n+1}J_{m+1}(p_1x_1)J_{n+1}(p_2x_2)\, dp_1\, dp_2 \int_0^\infty \int_0^\infty e^{-(p_1t_1+p_2t_2)}f(t_1, t_2)\, dt_1\, dt_2.$$

On changing the order of integration on the right hand side which is assumed permissible, and then integrating with respect to p_1 and p_2, we get

$$\int_0^\infty \int_0^\infty e^{-(x_1y_1+x_2y_2)}f(y_1, y_2)\, dy_1\, dy_2$$

$$= \frac{2^{m+1}2^{n+1}\Gamma(m+\frac{3}{2})\Gamma(n+\frac{3}{2})}{\pi} \int_0^\infty \int_0^\infty \frac{f(t_1, t_2)\, dt_1\, dt_2}{(t_1^2+x_1^2)^{m+3/2}\,(t_2^2+x_2^2)^{n+3/2}}$$

$$= \int_0^\infty \int_0^\infty \frac{f(t_1, t_2)\, dt_1\, dt_2}{t_1^{m}t_2^{n}} \int_0^\infty \int_0^\infty e^{-(x_1y_1+x_2y_2)}J_m(y_1t_1)J_n(y_2t_2)y_1^{m+1}y_2^{n+1}\, dy_1\, dy_2$$

$$= \int_0^\infty \int_0^\infty e^{-(x_1y_1+x_2y_2)}y_1^{m+\frac{1}{2}}y_2^{n+\frac{1}{2}}\, dy_1\, dy_2$$

$$\times \int_0^\infty \int_0^\infty \sqrt{(y_1t_1)}\ \sqrt{(y_2t_2)}\ J_m(y_1t_1)\ J_n(y_2t_2)t_1^{-m-\frac{1}{2}}t_2^{-n-\frac{1}{2}}f(t_1, t_2)\, dt_1\, dt_2,$$

on changing the order of integration again, which is assumed to be permissible.

We therefore, have

$$\int_0^\infty \int_0^\infty e^{-(x_1y_1+x_2y_2)}[y_1^{m+\frac{1}{2}}y_2^{n+\frac{1}{2}}F(y_1, y_2)-f(y_1, y_2)]dy_1\, dy_2 = 0,$$

where

$$F(y_1, y_2) = \int_0^\infty \int_0^\infty \sqrt{(y_1 t_1)} \sqrt{(y_2 t_2)}\, J_m(y_1 t_1)\, J_n(y_2 t_2)\, t_1^{-M-\frac{1}{2}} t_2^{-n-\frac{1}{2}} f(t_1, t_2)\, dt_1\, dt_2. \quad (9)$$

Hence assuming that $f(y_1, y_2)$ and $y_1^{m+\frac{1}{2}} y_2^{n+\frac{1}{2}} F(y_1, y_2)$ are continuous functions of both variables in $0 \leqq y_1 \leqq \infty$, $0 \leqq y_2 \leqq \infty$, by Lemma 3, we get

$$F(y_1, y_2) \equiv y_1^{-m-\frac{1}{2}} y_2^{-n-\frac{1}{2}} f(y_1, y_2),$$

which proves the proposition.

Theorem L. *Conversely, let*

$$\varphi(p_1, p_2) = p_1 p_2 \int_0^\infty \int_0^\infty e^{-(p_1 x_1 + p_2 x_2)} f(x_1, x_2)\, dx_1\, dx_2$$

and further let $x_1^{-m-\frac{1}{2}} x_2^{-n-\frac{1}{2}} f(x_1, x_2)$ be self-reciprocal in the Hankel Transform of orders m and n with respect to p_1 and p_2 respectively, then $p_1^{m-\frac{1}{2}} p_2^{n-\frac{1}{2}} \varphi(p_1, p_2)$ is self-reciprocal in the Hankel Transform of order $m+1$ and $n+1$ with respect to p_1 and p_2 respectively, provided the various changes in the orders of integrations performed in the course of the proof are permissible and the integral for the self-reciprocal property exists.

We have, by hypothesis,

$$x_1^{-m-\frac{1}{2}} x_2^{-n-\frac{1}{2}} f(x_1, x_2)$$
$$= \int_0^\infty \int_0^\infty \sqrt{(x_1 y_1)} \sqrt{(x_2 y_2)}\, J_m(x_1 y_1)\, J_n(x_2 y_2)\, y_1^{-m-\frac{1}{2}} y_2^{-n-\frac{1}{2}} f(y_1, y_2)\, dy_1\, dy_2.$$

We then get

$$\begin{aligned}
\varphi(p_1, p_2) &= p_1 p_2 \int_0^\infty \int_0^\infty e^{-(p_1 x_1 + n_2 x_2)} f(x_1, x_2)\, dx_1\, dx_2 \\
&= p_1 p_2 \int_0^\infty \int_0^\infty e^{-(p_1 x_1 + p_2 x_2)} x_1^{m+\frac{1}{2}} x_2^{n+\frac{1}{2}} f(x_1, x_2) x_1^{-m-\frac{1}{2}} x_2^{-n-\frac{1}{2}} dx_1\, dx_2 \\
&= p_1 p_2 \int_0^\infty \int_0^\infty e^{-(p_1 x_1 + p_2 x_2)} x_1^{m+\frac{1}{2}} x_2^{n+\frac{1}{2}}\, dx_1\, dx_2 \\
&\quad \times \int_0^\infty \int_0^\infty \sqrt{(x_1 y_1)} \sqrt{(x_2 y_2)}\, J_m(x_1 y_1)\, J_n(x_2 y_2)\, y_1^{-m-\frac{1}{2}} y_2^{-n-\frac{1}{2}} f(y_1, y_2)\, dy_1\, dy_2 \\
&= p_1 p_2 \int_0^\infty \int_0^\infty y_1^{-m} y_2^{-n} f(y_1, y_2)\, dy_1\, dy_2 \\
&\quad \times \int_0^\infty \int_8^\infty e^{-(p_1 x_1 + p_2 x_2)}\, J_m(x_1 y_1)\, J_n(x_2 y_2)\, x_1^{m+1}\, x_2^{n+1}\, dx_1\, dx_2,
\end{aligned}$$

on changing the order of integration, which is assumed permissible,

$$= \frac{2^{m+n+2}\Gamma(m+\frac{3}{2})\Gamma(n+\frac{3}{2})}{\pi} p_1^2 p_2^2 \int_0^\infty \int_0^\infty \frac{f(y_1, y_2)\, dy_1\, dy_2}{(p_1^2 + y_1^2)^{m+3/2} (p_2^2 + y_2^2)^{n+3/2}}.$$

Hence we have

$$\int_0^\infty\int_0^\infty \sqrt{(p_1z_1)}\sqrt{(p_2z_2)}J_{m+1}(p_1z_1)J_{n+1}(p_2z_2)p_1^{m-\frac{1}{2}}p_2^{n-\frac{1}{2}}\varphi(p_1,\, p_2)\,dp_1\,dp_2$$

$$= \frac{2^{m+n+2}\Gamma(m+\frac{3}{2})\Gamma(n+\frac{3}{2})\sqrt{(z_1z_2)}}{\pi}\int_0^\infty\int_0^\infty p_1^{m+2}p_2^{n+2}J_{m+1}(p_1z_1)J_{n+1}(p_2z_2)dp_1dp_2$$

$$\times\int_0^\infty\int_0^\infty \frac{f(y_1,\, y_2)\,dy_1\,dy_2}{(p_1^2+y_1^2)^{m+3/2}(p_2^2+y_2^2)^{n+3/2}}$$

$$= \frac{2^{m+n+2}\Gamma(m+\frac{3}{2})\Gamma(n+\frac{3}{2})\sqrt{(z_1z_2)}}{\pi}$$

$$\times\int_0^\infty\int_0^\infty f(y_1,\, y_2)dy_1\,dy_2\int_0^\infty\int_0^\infty \frac{p_1^{m+2}J_{m+1}(p_1z_1)p_2^{n+2}J_{n+1}(p_2z_2)\,dp_1\,dp_2}{(p_1^2+y_1^2)^{m+3/2}(p_2^2+y_2^2)^{n+3/2}},$$

on changing the order of integration again, which is also assumed permissible,

$$= z^{m+\frac{1}{2}}z^{n+\frac{1}{2}}\int_0^\infty\int_0^\infty e^{-(y_1z_1+y_2z_2)}f(y_1,\, y_2)dy_1\,dy_2$$

$$= z_1^{m-\frac{1}{2}}z_2^{n-\frac{1}{2}}\varphi(z_1,\, z_2)$$

Theorems K and L are extensions of theorems I and II (Mitra and Bose, 1958)

Next we prove the following theorem and its converse.

Theorem M. *Let* $\varphi(p_1, p_2)/p_1p_2$ *be self reciprocal in the cosine transform with respect to* p_1 *and* p_2, *then* $f(x_1, x_2)$ *is self-reciprocal in the sine transform with respect to both* x_1 *and* x_2, *provided* $f(x_1, x_2)$ *is continuous in both the variables in* $0 \leqq x_i \leqq \infty$, $i = 1, 2$ *and so is also* $F(x_1, x_2)$, *which is defined below in* (10).

At the outset we note as before that the various changes in the orders of integrations performed in the course of establishing the theorem are supposed to be justifiable.

By hypothesis, we have

$$\frac{\varphi(p_1,\, p_2)}{p_1p_2} = \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\varphi(y_1,\, y_2)}{y_1y_2}\cos p_1y_1\cos p_2y_2\,dy_1\,dy_2.$$

Employing the relation

$$\varphi(y_1,\, y_2) = y_1y_2\int_0^\infty\int_0^\infty e^{-(y_1x_1+y_2x_2)}f(x_1,\, x_2)\,dx_1\,dx_2$$

and then changing the order of integrations, we get

$$\frac{\varphi(p_1,\, p_2)}{p_1\,p_2} = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(x_1,\, x_2)\,dx_1\,dx_2\int_0^\infty\int_0^\infty e^{-(y_1x_1+y_2x_2)}\cos p_1y_1\cos p_2y_2\,dy_1\,dy_2$$

$$= \frac{2}{\pi}\int_0^\infty\int^\infty \frac{f(x_1,\, x_2)\,x_1x_2\,dx_1\,dx_2}{(x_1^2+p_1^2)(x_2^2+x_2^2)}$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^\infty f(x_1, x_2)\,dx_1\,dx_2\int_0^\infty\int_0^\infty e^{-(p_1y_1+p_2y_2)}\sin x_1y_1\sin x_2y_2\,dy_1\,dy_2$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^\infty e^{-(p_1x_1+p_2x_2)}\,dx_1\,dx_2\int_0^\infty\int_0^\infty f(y_1, y_2)\sin x_1y_1\sin x_2y_2\,dy_1\,dy_2$$

on writing x_1 for y_1 and x_2 for y_2 and vice versa and then changing the order of integration,

$$= \int_0^\infty\int_0^\infty e^{-(p_1x_1+p_2x_2)}F(x_1, x_2)\,dx_1\,dx_2,$$

where

$$F(x_1, x_2) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(y_1, y_2)\sin x_1y_1\sin x_2y_2\,dy_1\,dy_2 \tag{10}$$

We, therefore, get

$$\int_0^\infty\int_0^\infty e^{-(p_1x_1+p_2x_2)}[f(x_1, x_2)-F(x_1, x_2)]dx_1\,dx_2 = 0$$

Hence by Lemma 3, we have $f(x_1, x^5)\equiv F(x_1, x_2)$, which proves the theorem.

Theorem N. *Conversely, let* $f(x_1, x_2)$ *be self-reciprocal in the sine transform with respect to both* x_1 *and* x_2, *then* $\varphi(p_1, p_2)/p_1p_2$ *is self-reciprocal in the cosine transform with respect to both* p_1 *and* p_2, *provided the integral for self-reciprocal property exists.*

We have

$$f(x_1, x_2) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(y_1, y_2)\sin x_1y_1\sin x_2y_2\,dy_1\,dy_2\,;$$

then making use of this result in the relation

$$\frac{\varphi(p_1, p_2)}{p_1\,p_2} = \int_0^\infty\int_0^\infty e^{-(p_1x_1+p_2x_2)}f(x_1, x_2)\,dx_1\,dx$$

and on changing the order of integrations which is supposed permissible, we get

$$\frac{\varphi(p_1, p_2)}{p_1\,p_2} = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(y_1, y_2)\,dy_1\,dy_2\int_0^\infty\int_0^\infty e^{(p_1x_1+p_2x_2)}\sin x_1y_1\sin x_2y_2\,dy_1\,dy_2$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{f(y_1, y_2)\,y_1\,y_2\,dy_1\,dy_2}{(p_1^2+y_1^2)(p_2^2+y_2^2)}.$$

Therefore we have

$$\frac{2}{\pi}\int_0^\infty\int_0^\infty \frac{\varphi(p_1, p_2)}{p_1\,p_2}\cos p_1\,x_1\cos p_2\,x_2\,dp_1\,dp_2$$

$$=\left(\frac{2}{\pi}\right)^2\int_0^\infty\int_0^\infty \cos p_1\,x_1\cos p_2\,x_2\,dp_1\,dp_2\int_0^\infty\int_0^\infty \frac{f(y_1, y_2)\,y_1\,y_2\,dy_1\,dy_2}{(p_1^2+y_1^2)\,(p_2^2+y_2^2)}$$

$$=\left(\frac{2}{\pi}\right)^2\int_0^\infty\int_0^\infty f(y_1, y_2)y_1\,y_2\,dy_1\,dy_2\int_0^\infty\int_0^\infty \frac{\cos p_1\,x_1\cos p_2\,x_2\,dp_1\,dp_2}{(p_1^2+y_1^2)(p_2^2+y_2^2)},$$

on changing the order of integrations again, which is assumed permissible,

$$= \int_0^\infty \int_0^\infty {}^{-(x_1 y_1 + x_2 y_2)} f(y_1, y_2)\, dy_1\, dy_2 = \frac{\varphi(x_1, x_2)}{x_1 x_2}.$$

These two theorems M and N are the extensions of theorems III and IV (Bose, 1952) to two variables.

By proceeding in the above way we can also make similar extensions of theorems V and VI, (Mitra and Bose, 1953).

2. We next proceed to extend the results of Theorems C and D to two variables. For this we first establish the following :

Lemma 4. *Let* $G(y_1, y_2)$ *be bounded and continuous in both variables in* $0 \leqq y_i \leqq \infty$, $i = 1, 2$ *and further* $\int_0^\infty | G(y_1, y_2) | dy_i$ *exist and be integrable in* $(0, \infty)$

If also $G(y_1, y_2)$ *be such that*

$$\int_0^\infty \int_0^\infty \frac{G(y_1, y_2)\, dy_1\, dy_2}{y_1^m + a^m)^n (y_2^m + b^m)^n} = 0$$

where $a, b > 0$, $m, n > 1$, *then* $G(y_1, y_2) \equiv 0$.

First, let

$$H(y_2) = \int_0^\infty \frac{G(y_1, y_2)\, dy_1}{(y_1^m + a^m)^n}$$

Then $G(y_1, y_2)$ being bounded, the integral on the right hand side is uniformly convergent with respect to y_2 in $(0, \infty)$. It, therefore, follows that $H(y_2)$ is continuous in $(0, Y)$, where Y is arbitrary and hence $H(y_2)$ is bounded and integrable in $(0, Y)$.

Further

$$\int_{Y_2'}^{Y_2''} | H(y_2) | dy_2 \leqq \int_{Y_2'}^{Y_2''} dy_2 \int_0^\infty \frac{| G(y_1, y_2) | dy_1}{(y_1^m + a^m)^n}.$$

Changing the order of integration which is obviously permissible, we easily get

$$\int_{Y_2'}^{Y_2''} | H(y_2) | dy_2 \leqq \int_0^\infty \frac{dy_1}{(y_1^m + a^m)^n} \int_{Y_2'}^{Y_2''} | G(y_1\, y_2) | dy_2 < k \int_{Y_2'}^{Y_2''} | G(y_1, y_2) | dy_2 < \varepsilon,$$

whenever $Y_2'' > Y_2' \geqq Y_2$, say.

$H(y_2)$ is thus absolutely integrable in $(0, \infty)$.

Since

$$\int_0^\infty \frac{H(y_2)\, dy_2}{(y_2^m + b^m)^n} = 0,$$

hence by Lemma 2 we get $H(y_2) \equiv 0$.

Therefore we have

$$\int_0^\infty \frac{G(y_1, y_2)\, dy_1}{(y_1^m + a^m)^n} = 0$$

Hence again by Lemma 2, we get $G(y_1, y_2) \equiv 0$.

With the help of this Lemma we shall now prove

Theorem 0 : *Let*

$$\psi_1(p_1, p_2) = \int_0^\infty \int_0^\infty \surd(p_1 x_1) \surd(p_2 x_2)\, K_m(p_1 x_1)\, K_n(p_2 x_2)\, f(x_1, x_2\, dx_1\, dx_2$$

and furthe let $p_1^{2m}\, p_2^{2n} \psi_1(p_1, p_2)$ *be self-reciprocal in the Hankel Transform of orders* $3m$ *and* $3n$ *with respect to* p_1 *and* p_2 *respectively. Then* $x^{-2m}\, x^{-2n} f(x_1, x_2)$ *is self-reciprocal in the Hankel Transform of orders* m *and* n *with respect to* x_1 *and* x_2 *respectively, provided* $y_1^{-m+\frac{1}{2}}\, y_2^{-n+\frac{1}{2}}\, f(y_1, y_2)$ *and* $y_1^{m+\frac{1}{2}}\, y_2^{n+\frac{1}{2}}\, F(y_1, y_2)$ *are continuous in both variables in* $0 \leqq y_i \leqq \infty$, $i = 1, 2$ *and are absolutely integrable with respect to* y_1 *in* $(0, \infty)$ *and so also with respect to* y_2 *and further*

$$\int_0^\infty y^{-m+\frac{1}{2}} y_2^{-n+\frac{1}{2}}\, |f(y_1, y_2)|\, dy_i \text{ , } (i = 1, 2)$$

$$\int_0^\infty y^{m+\frac{1}{2}} y_2^{n+\frac{1}{2}}\, |F(y_1, y_2)|\, dy_i \text{ , } (i = 1, 2)$$

are integrable in $(0, \infty)$ *where* $F(y_1, y_2)$ *is defined below in* (11).

We have by hypothesis

$$\begin{aligned} & y_1^{2m} y_2^{2n} \psi_1(y_1, y_2) \\ &= \int_0^\infty \int_0^\infty \surd(p_1 y_1) \surd(p_2 y_2) J_{3m}(p_1 y_1) J_{3n}(p_2 y_2) p_1^{2m} p_2^{2n} \psi_1(p_1 p_2)\, dp_1\, dp_2 \\ &= \int_0^\infty \int_0^\infty \surd(p_1 y_1) \surd(p_2 y_2) J_{3m}(p_1 y_1) J_{3n}(p_2 y_2) p_1^{2m} p_2^{2n}\, dp_1\, dp_2 \\ &\qquad \times \int_0^\infty \int_0^\infty \surd(p_1 x_1) \surd(p_2 x_2) K_m(p_1 x_1) K_n(p_2 x_2) f(x_1, x_2)\, dx_1\, dx_2 \\ &= \int_0^\infty \int_0^\infty \surd(x_1 y_1) \surd(x_2 y_2) f(x_1, x_2)\, dx_1\, dx_2 \\ &\qquad \times \int_0^\infty \int_0^\infty p_1^{2m+1} p_2^{2n+1} J_{3m}(p_1 y_1) K_m(p_1 x_1) J_{3n}(p_2 y_2) K_n(p_2 x_2)\, dp_1\, dp_2, \end{aligned}$$

on changing the order of integration which is assumed permissible,

$$= \Gamma(2m+1)\Gamma(2n+1) 2^{2m} 2^{2n} y_1^{3m+\frac{1}{2}} y_2^{3n+\frac{1}{2}} \int_0^\infty \int_0^\infty \frac{x_1^{\frac{1}{2}-m} x_2^{\frac{1}{2}-n} f(x_1, x_2)\, dx_1\, dx_2}{(x_1^2 + y_1^2)^{2m+1} (x_2^2 + y_2^2)^{2n+1}} .$$

Hence, on writing p_1 for y_1 and p_2 for y_2 and y_1 and y_2 for x_1 and x_2 respectively, we get

$$\psi_1(p_1, p_2) = 2^{2m} 2^{2n} \Gamma(2m+1)\Gamma(2n+1) p_1^{m+\frac{1}{2}} p_2^{n+\frac{1}{2}} \int_0^\infty \int_0^\infty \frac{y_1^{\frac{1}{2}-m} y_2^{\frac{1}{2}-n} f(y_1, y_2)\, dy_1\, dy_2}{(y_1^2 + p_1^2)^{2m+1} (y_2^2 + p_2^2)^{2n+1}} .$$

Again

$$\psi_1(p_1, p_2) = 2^{2m}2^{2n}\Gamma(2m+1)\Gamma(2n+1)p_1^{m+\frac{1}{2}}p_2^{n+\frac{1}{2}}$$

$$\times \int_0^\infty \int_0^\infty x_1^{\frac{1}{2}-2m}x_2^{\frac{1}{2}-2n}f(x_1, x_2)\, dx_1\, dx_2 \int_0^\infty \int_0^\infty \frac{y_1^{m+1}y_2^{n+1}J_m(x_1 y_1)J_n(x_2 y_2)\, dy_1\, dy_2}{(p_1^2+y_1^2)^{2m+1}(p_2^2+y_4^2)^{2n+1}}$$

$$= 2^{2m}2^{2n}\Gamma(2m+1)\Gamma(2n+1)p_1^{m+\frac{1}{2}}p_2^{n+\frac{1}{2}} \int_0^\infty \int_0^\infty \frac{y_1^{m+\frac{1}{2}}\, y_2^{n+\frac{1}{2}}\, dy_1\, dy_2}{(p_1^2+y_1^2)^{2m+1}(p_2^2+y_2^2)^{2n+1}}$$

$$\times \int_0^\infty \int_0^\infty \sqrt{(x_1y_2)}\, \sqrt{(x_1y_2)}J_m(x_1x_1)\, J_n(x_2x_2)\, x_1^{-2m}x_2^{-2n}\, f(x_1, x_2)\, dx_1\, dx_2$$

on changing the order of integration again, which is assumed permissible.

Hence we get

$$\int_0^\infty \int_0^\infty \frac{y_1^{\frac{1}{2}-m}y_2^{\frac{1}{2}-n}f(y_1, y_2)\, dy_1\, dy_2}{(y_1^2+p_1^2)^{2m+1}(y_2^2+p_2^2)^{2n+1}} = \int_0^\infty \int_0^\infty \frac{y_1^{m+\frac{1}{2}}y_2^{n+\frac{1}{2}}F(y_1, y_2)\, dy_1\, dy_2}{(y_1^2+p_1^2)^{2m+1}(y_2^2+p_2^2)^{2n+1}},$$

where

$$F(y, y_2) = \int_0^\infty \int_0^\infty \sqrt{(x_1y_1)}\, \sqrt{(x_2y_2)}J_m(x_1y_1)J_n(x_2y_2)x_1^{-2m}x_2^{-2n}f(x_1, x_2)\, dx_1\, dx_2\,; \qquad (11)$$

that is

$$\int_0^\infty \int_0^\infty \frac{[y_1^{\frac{1}{2}-m}y_2^{\frac{1}{2}-n}f(y_1, y_2)-y_1^{m+\frac{1}{2}}y_2^{n+\frac{1}{2}}F(y_1, y_2)]\, dy_1\, dy_2}{(y_1^2+p_1^2)^{2m+1}(y_2^2+p_2^2)^{2n+1}} = 0$$

Hence, provided $y_1^{\frac{1}{2}-m}y_2^{\frac{1}{2}-n}f(y_1, y_2)$ and $y_1^{m+\frac{1}{2}}y_2^{n+\frac{1}{2}}F(y_1, y_2)$ and continuous functions of both the variables in $0 \leqq y_i \leqq \infty$ and other conditions stated above hold good, we get, by Lemma 4,

$$F(y_1, y_2) \equiv y_1^{-2m}y_2^{-2n}f(y_1, y_2),$$

which proves the proposition..

Theorem P. *Conversely, let* $x_1^{-2m}x_2^{-2n}f(x_1, x_2)$ *be self-reciprocal in the Hankel transform of orders* m *and* n *with respect to* x_1 *and* x_2 *respectively, then* $p_1^{2m}p_2^{2n}\, \psi_1(p_1, p_2)$ *is self-reciprocal in the Hankel transform of orders* 3m *and* 3n *with respect to* p_1 *and* p_2 *respectively, provided the integral for self-reciprocal property holds good*

Proceeding as before we get

$$\psi_1(p_1, p_2) = 2^{2m}2^{2n}\Gamma(2m+1)\Gamma(2n+1)p_1^{m+\frac{1}{2}}p_2^{n+\frac{1}{2}} \int_0^\infty \int_0^\infty \frac{y_1^{m+\frac{1}{2}}y_2^{n+\frac{1}{2}}F(y_1, y_2)\, dy_1\, dy_2}{(p_1^2+y_1^2)^{2m+1}(p_2^2+y_2^2)^{2n+1}}$$

But by hypothesis, we have

$$F(y_1, y_2) = y_1^{-2m}y_2^{-2n}f(y_1, y_2)$$

Hence we get

$$\psi_1(p_1, p_2) = 2^{2m}2^{2n}\Gamma(2m+1)\Gamma(2n+1)p_1^{m+\frac{1}{2}}p_2^{n+\frac{1}{2}} \int_0^\infty \int_0^\infty \frac{y_1^{\frac{1}{2}-m}y_2^{\frac{1}{2}-n}f(y_1, y_2)\, dy_1\, dy_2}{(p_1^2+y_1^2)^{2m+1}(p_2^2+y_2^2)^{2n+1}}$$

We get also by proceeding as before

$$\int_0^\infty \int_0^\infty \sqrt{(p_1x_1)}\sqrt{(p_2x_2)}J_{3m}(p_1x_1)J_{3n}(p_2x_2)p_1^{2m}p_2^{2n}\psi_1(p_1, p_2)\,dp_1\,dp_2$$

$$= 2^{2m}2^{2n}\Gamma(2m+1)\Gamma(2n+1)x_1^{3m+\frac{1}{2}}x_2^{3n+\frac{1}{2}}\int_0^\infty\int_0^\infty \frac{y_1^{\frac{1}{2}-m}y_2^{\frac{1}{2}-n}f(y_1, y_2)\,dy_1\,dy_2}{(x_1^2+y_1^2)^{2m+1}(x_2^2+y_2^2)^{2n+1}}$$

$$= \frac{\psi_1(x_1, x_2)}{x_1^{m+\frac{1}{2}}x_2^{n+\frac{1}{2}}}\,x_1^{3m+\frac{1}{2}}x_2^{3n+\frac{1}{2}}$$

$$= x_1^{2m}x_2^{2n}\,\psi_1(x_1, x_2).$$

It is worthwhile to observe here that the theorem holds good, provided the various changes in the orders of integrations are justifiable.

Proceeding in a similar way we can establish the extensions of Theorems E, F, G, H and also of Theorems A and B to two variables.

In conclusion I wish to express my grateful thanks to Dr. S. C. Mitra for his kind help in the preparation of this paper.

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TURBULENT FLOW IN A PLANE WAKE OF A COMPRESSIBLE FLUID

By

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1. Introduction. The problems of turbulent flow in a compressible fluid have been studied by some authors. Abramovich considered the case of the mixing of a parallel stream with the adjacent medium at rest and Görtler (Görtler, 1942) re-examined the same problem and obtained some improvement in the velocity profiles on assumption, due to Reichardt, that the exchange coefficient is constant over each cross-section of the mixing zone for free turbulence. Pai (Pai, 1949) took up the problem of plane jet exhausting into uniform stream and of the mixing of two uniform streams. Recently he (Pai, 1952) extended the case to the problem of an axially symmetrical jet.

In this paper, the problem of a plane wake in a compressible fluid has been studied, using both Prandtl's mixture length theory and Reichardt's theory of free turbulence. The fundamental equation of motion has been simplified under the assumption of small perturbation and the variables have been transformed by Mises' transformation. For free turbulence Prandtl's number has been taken to be unity, so that the energy equation gives at once Crocco's relation for the temperature distribution.

It is found that when the Prandtl's assumption regarding the constancy of the mixture length over any section of the wake is adopted, the solution gives a finite breadth to the wake; whereas when the Reichardt's assumption regarding the constancy of the exchange coefficient over any section is taken, the breadth of the wake becomes infinite. It is also found that when the velocity and temperature distributions are identical, the case degenerates to the corresponding problem for the incompressible wake.

2. Basic Equations. Very far down-stream in the wake, if the motion is steady boundary layer approximations hold good. The pressure would be nearly constant across a section of the wake, the transverse velocity small compared with the longitudinal velocity. The axis of x is taken along the axis of the wake with the origin at some unspecified point in the neighbourhood of the cylindrical obstacle. Let u_0 be the undisturbed stream velocity and (u, v) the components of mean velocity in the wake.

The equation of motion in the x-direction is

$$\varrho u\frac{\partial u}{\partial x}+\varrho v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\varepsilon\varrho\frac{\partial u}{\partial y}\right), \tag{1}$$

where ϱ is the mean density and ε is the exchange coefficient of the turbulent flow.

The equation of energy is

$$\varrho u\frac{\partial i}{\partial x}+\varrho v\frac{\partial i}{\partial y} = \frac{\partial}{\partial y}\left(\varepsilon\varrho\frac{\partial i}{\partial y}\right)+\varepsilon\varrho\left(\frac{\partial u}{\partial y}\right)^3, \tag{2}$$

where i is the heat-content.

The equation of continuity is

$$\frac{\partial}{\partial x}(\varrho u)+\frac{\partial}{\partial y}(\varrho v) = 0, \tag{3}$$

and the equation of state for a perfect gas is

$$i\varrho = \text{const} = i_0\varrho_0, \tag{4}$$

where suffix 0 denotes the value in the undisturbed stream

The boundary conditions are

(a) on the axis of the wake $(y = 0)$,

$$v = 0, \tag{5}$$

and

$$\frac{\partial u}{\partial y} = 0, \tag{6}$$

(b) at the edge of the wake, $u = u_0$. (7)

3. **Simplifications of the equations.** According to the mixture length theory, the Prandtl number for free turbulence is equal to unity. The equations (1) and (2) are satisfied by Crocco's relation

$$i = A_1+A_2u-u^2/2$$

or

$$i_0-i = c(u_0-u)-\tfrac{1}{2}(u_0^2-u^2). \tag{8}$$

Let us now put

$$u = u_0(1-\omega), \tag{9}$$

where ω is small compared to u_0.

Neglecting squares and products of small quantities, the equation (1) reduces to

$$\varrho u_0\frac{\partial\omega}{\partial x} = \frac{\partial}{\partial y}\left(\varepsilon\varrho\frac{\partial\omega}{\partial y}\right). \tag{10}$$

The equation (3) is satisfied by

$$\varrho u = \frac{\partial\psi}{\partial y},\ \varrho v = -\frac{\partial\psi}{\partial x}. \tag{11}$$

We now change the independent variables (x, y) to (x, ψ) by means of the transformation formulae

$$\left.\begin{aligned}\left(\frac{\partial}{\partial y}\right)_x &= \varrho u\left(\frac{\partial}{\partial\psi}\right)_x = \varrho u_0(1-\omega)\left(\frac{\partial}{\partial\psi}\right)_x,\\ \left(\frac{\partial}{\partial x}\right)_y &= -\varrho v\left(\frac{\partial}{\partial\psi}\right)_x+\left(\frac{\partial}{\partial x}\right)_\psi.\end{aligned}\right\} \tag{12}$$

The equation (10) then becomes to our approximation,

$$\frac{\partial\omega}{\partial x} = u_0\frac{\partial}{\partial\psi}\left(\varepsilon\varrho^3\frac{\partial\omega}{\partial\psi}\right). \tag{13}$$

Now according to Prandtl's theory

$$\varepsilon = l^2\left|\frac{\partial u}{\partial y}\right|, \tag{14}$$

where l is the mixture length which is assumed to be constant over any one section of the wake. But u is least so that ω is greatest in the middle of the wake ($y = 0$), hence when y is positive, $\frac{\partial\omega}{\partial y}$ is negative and therefore

$$\varepsilon = -l^2u_0\frac{\partial\omega}{\partial y}$$

$$= -l^2u_0^2\varrho(1-\omega)\frac{\partial\omega}{\partial\psi}. \tag{15}$$

Take

$$l^2 = \alpha x, \tag{16}$$

where α is a constant.

Then the equation (13) comes to our approximation,

$$\frac{\partial\omega}{\partial x} = -\alpha u_0^3x\frac{\partial}{\partial\psi}\left\{\varrho^3\left(\frac{\partial\omega}{\partial\psi}\right)^2\right\}. \tag{17}$$

We can now replace ϱ^3 in the right hand side of equation (17) by ϱ_0^3, as this will not harm our approximation. Hence we get

$$\frac{\partial\omega}{\partial x} = -2\alpha xu_0^3\varrho_0^3\frac{\partial\omega}{\partial\psi}\frac{\partial^2\omega}{\partial\psi^2}. \tag{18}$$

To solve this equation, let us put

$$\omega = x^{-\frac{1}{2}}f(\eta) \tag{19}$$

where

$$\eta = Bx^{-\frac{1}{2}}\psi. \tag{20}$$

We then get from (18)

$$f(\eta)+\eta f'(\eta) = 4\alpha f'(\eta)f''(\eta), \tag{21}$$

when we take

$$B = \frac{1}{u_0\varrho_0}. \tag{22}$$

Before we solve (21), we must find out the conditions to be satisfied by $f(\eta)$.

4. Solution of the equations. From definition, we have

$$d\psi = \left(\frac{\partial\psi}{\partial x}\right)_\eta dx+\left(\frac{\partial\psi}{\partial\eta}\right)_x d\eta = \left(\frac{\partial\psi}{\partial y}\right)_x dy+\left(\frac{\partial\psi}{\partial x}\right)_y dx$$

where

$$\psi = u_0\varrho_0x^{\frac{1}{2}}\eta.$$

Therefore $$\tfrac{1}{2}u_0\varrho_0 x^{-\frac{1}{2}}\eta\,dx + u_0\varrho_0 x^{-\frac{1}{2}}d\eta = \varrho u\,dy - \varrho v\,dx,$$

giving $$\varrho v = -\tfrac{1}{2}u_0\varrho_0 x^{-\frac{1}{2}}\eta \tag{23}$$

and $$\varrho u\,dy = u_0\varrho_0 x^{\frac{1}{2}}d\eta, \tag{24}$$

From (23) we see that $v = 0$ when $\eta = 0$, thus $\eta = 0$ corresponds to $y = 0$.

From (24) we get

$$dy = \frac{u_0\varrho_0}{u\varrho}x^{\frac{1}{2}}d\eta$$

$$= \frac{u_0\varrho_0}{i_0u_0\varrho_0}\,\frac{i}{1-\omega}\,x^{\frac{1}{2}}d\eta, \text{ from (4) and (9).} \tag{25}$$

But from (8) and (9) we have approximately

$$i_0 - i = u_0(c-u_0)\omega. \tag{26}$$

Hence $$dy = \{1-(u_0/i_0)(c-u_0)\omega+\omega\}x^{\frac{1}{2}}d\eta$$

or $$dy = \{x^{\frac{1}{2}}-(u_0/i_0)(c-u_0)f(\eta)+f(\eta)\}d\eta. \tag{27}$$

Hence $$y = x^{\frac{1}{2}}\eta - \left\{\frac{u_0}{i_0}(c-u_0)-1\right\}\int_0^{\eta} f(\eta)d\eta, \tag{28}$$

as $y = 0$ when $\eta = 0$.

Further $$\frac{\partial\omega}{\partial y} = \varrho u\frac{\partial\omega}{\partial\psi} = \frac{\varrho u}{\varrho_0u_0}x^{-1}f'(\eta), \tag{29}$$

so that the condition $\dfrac{\partial u}{\partial y} = 0$ when $y = 0$ or $\dfrac{\partial\omega}{\partial y} = 0$ when $y = 0$ reduces to the condition $f'(\eta) = 0$ when $\eta = 0$.

Thus the boundary conditions are

(a) on $$\eta = 0,\quad f'(\eta) = 0, \tag{30}$$

(b) at the edge of the wake, $$f(\eta) = 0. \tag{31}$$

We are now in a position to solve equation (21). The solution corresponding to the conditions (30) and (31) is

$$f(\eta) = \frac{\eta_0^{\,3}}{18\alpha}\left\{1-\left(\frac{\eta}{\eta_0}\right)^{3/2}\right\}^2, \tag{32}$$

so that $\eta = \eta_0$ corresponds to the edge of the wake.

With this value of $f(\eta)$, we have from (28),

$$y = x^{\frac{1}{2}}\eta - \left\{\frac{u_0}{i_0}(c-u_0)-1\right\}\frac{\eta_0^{\,3}\eta}{18\alpha}\left\{1-\tfrac{4}{5}\left(\frac{\eta}{\eta_0}\right)^{3/2}+\tfrac{1}{4}\left(\frac{\eta}{\eta_0}\right)^3\right\}. \tag{33}$$

If D be the drag per unit breadth of the cylindrical obstacle, we have

$$D = 2\int_0^{\eta_0}\varrho u(u_0-u)dy$$

$$= 2\int_0^{\eta_0} u_0^2 \varrho_0 \omega x^{\frac{1}{2}} dy \qquad \text{from (24)}$$

$$= \frac{u_0^2 \varrho_0 \eta_0^3}{9\alpha} \int_0^{\eta_0} \left\{1-\left(\frac{\eta}{\eta_0}\right)^{3/2}\right\}^2 d\eta = \frac{u_0^2 \varrho_0 \eta_0^4}{20\alpha}. \tag{34}$$

If $-2Q$ be the rate of transfer of heat from the fluid to the cylinder we have

$$-2Q = 2\int_0^{\eta_0} \varrho u(\iota_0 - \iota)\, dy$$

$$= 2u_0^2(c-u_0)\varrho_0 \int_0^{\eta_0} x^{\frac{1}{2}} \omega \, d\eta \qquad \text{from (26) and (24)}$$

$$= \frac{u_0^2 \varrho_0 (c-u_0) \eta_0^4}{2\alpha}.$$

Hence from (33) we get

$$y = x^{\frac{1}{2}}\eta + \frac{20}{9}\left(\frac{D}{2\varrho_0 u_0^2} + \frac{Q}{u_0 \varrho_0 \iota_0}\right)\left(\frac{\eta}{\eta_0}\right)\left\{1 - \tfrac{4}{5}\left(\frac{\eta}{\eta_0}\right)^{3/2} + \tfrac{1}{4}\left(\frac{\eta}{\eta_0}\right)^3\right\}. \tag{36}$$

Thus the edge of the wake is given by

$$y_0 = x^{\frac{1}{2}}\eta_0 + \frac{D}{2\varrho_0 u_0^2} + \frac{Q}{\varrho_0 u_0 \iota_0}. \tag{37}$$

Thus Prandtl's assumptions (14) and (16) give a finite breadth to the wake with a parabolic profile. The profile however does not start from the axis of the wake, but at some distance from the axis, depending on the dimension of the cylindrical obstacle.

5. **Reichardt's theory.** According to Reichardt's assumption, the exchange coefficient ε is constant over any one section of the wake.

We take here $\qquad \varepsilon = \alpha x^n, \qquad$ (14a)

where α is a constant.

Then the equation (13) becomes, replacing ϱ by ϱ_0 on the right hand side,

$$\frac{\partial \omega}{\partial x} = \alpha u_0 \varrho_0^3 x^n \frac{\partial^2 \omega}{\partial \psi^2}.$$

Putting $X = \dfrac{x^{n+1}}{n+1}$, the above equation becomes

$$\frac{\partial \omega}{\partial X} = \alpha u_0 \varrho_0^3 \frac{\partial^2 \omega}{\partial \psi^2}. \tag{18a}$$

We again put

$$\omega = X^{-\frac{1}{2}} f(\eta), \tag{19a}$$

$$\eta = B X^{-\frac{1}{2}} \psi, \tag{20a}$$

then (18a) gives

$$f(\eta) + \eta f'(\eta) = -\tfrac{1}{2} f''(\eta), \tag{21a}$$

with $$B = 1/(2\varrho_0\alpha^{\frac{1}{2}}u_0^{\frac{1}{2}}), \tag{22a}$$

Solution of (21a) is $$f(\eta) = \exp(-\eta^2) \tag{32a}$$

satisfying the conditions $$f'(\eta) = 0 \quad \text{when } \eta = 0$$

and $$f(\eta) = 0 \quad \text{when } \eta = \infty.$$

Thus the edge of the wake corresponds to $\eta = \infty$

Proceeding as before, we have in this case

$$y = 2\left(\frac{\alpha}{u_0(n+1)}\right)^{\frac{1}{2}} x^{(n+1)/2}\eta - 2\left(\frac{\alpha}{u_0}\right)^{\frac{1}{2}}\left\{\frac{u_0}{i_0}(c-u_0)-1\right\}\int_0^{\eta} \exp(-\eta^2)\,d\eta. \qquad \text{(33a}$$

Also we have $$D = 2\varrho_0 u_0^2(\pi\alpha/u_0)^{\frac{1}{2}}, \tag{34a}$$

and $$-2Q = 2\varrho_0 u_0(c-u_0)(\pi\alpha u_0)^{\frac{1}{2}}. \tag{35a}$$

Hence $$y = 2\left(\frac{\alpha}{u_0(n+1)}\right)^{\frac{1}{2}} x^{(n+1)/2}\eta + \left(\frac{D}{2\varrho_0 u_0^2} + \frac{Q}{\varrho_0 u_0 i_0}\right)\mathrm{erf}(\eta). \tag{36a}$$

Thus Reichardt's theory gives infinite breadth to the wake.

6. Incompressible liquid. If the velocity distribution be similar to the temperature distribustion,

i e. $$i/i_0 = u/u_0 = 1-\omega,$$

then from (26) we get $$(u_0/i_0)(c-u_0) = 1.$$

In this case the equation (33) gives

$$y = x^{\frac{1}{2}}\eta \tag{38}$$

and (33a) gives $$y = 2\{\alpha/u_0(n+1)\}^{\frac{1}{2}}x^{(n+1)/2}\eta.$$

The result (38) corresponds to the case of incompressible liquid with Prandtl's assumption.

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NOTE ON THE POSSIBILITY OF POLYTROPIC STARS WITH POWER LAW FOR OPACITY AND SUBATOMIC ENERGY GENERATION *

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Introduction : It is a very common practice with astrophysicists to consider stars in which the opacity, κ and the rate of generation of subatomic energy, ε obey laws given by the equations

$$\kappa = K \cdot \varrho^{\alpha} T^{\nu} \tag{1A}$$

$$\varepsilon = C \cdot \varrho^{b} T^{w}, \tag{1B}$$

where K and C are constants for a particular star but depend upon the stellar material. These laws are very suitable for homology considerations. Kramer's law of opacity is of form (1A) and Bethe's law of energy generation given by

$$\varepsilon \propto \varrho T^{-\frac{2}{3}} e^{-B/T^{\frac{1}{3}}}$$

can very well be approximated by form (1B) (Sinha, 1944). This proves the plausibility of the assumtions given by equations (1) The characteristics of such models, therefore, are worth investigating. This paper investigates under what condition (*i.e.* under what relation between α, ν, b, w) a model with law given by (1), can give rise to a polytropic star *i.e.* ($\varrho \propto T^n$, n finite) of finite mass and finite central temperature. The radiation pressure has been neglected in camparison with gas pressure. It should further be noted that the ordinary isothermal model does not come under our discussion, since n is finite.

Investigations. It is well known that in a star under radiative equilibrium (with spherical symmetry)

$$\frac{dT^4}{dP} = \frac{3}{4\pi acG} \cdot \kappa \cdot \frac{1}{m} \int_0^m \varepsilon \cdot dm \tag{2}$$

where T is the temperature, P the pressure and m the mass enclosed by a sphere passing through the point under consideration and having its centre at the centre of the star. The constants c and G stand for the velocity of light and the gravitational constant

Now using (1) and the well known relation

$$P = R\varrho T \tag{3}$$

and the assumed polytropic relation.

$$\varrho = A \cdot T^n \qquad (n, \text{ finite}) \tag{4}$$

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* The summary of this paper has been printed in the Calcutta Session (Jan. 1952) of the Indian Science Congress (Mathematics section). The conclusions have since been refined.

(A being a constant), we easily get

$$T^x = \lambda \frac{1}{m} \cdot \int_0^m T^y \cdot dm, \tag{5}$$

where

$$x = 3-\nu-(1+\alpha)n$$

$$y = w+nb$$

$$\lambda = \frac{3(n+1)RKCA^{\alpha+b+1}}{16\pi acG}. \tag{6}$$

We shall now show that $x = 0$ and $y = 0$ is the only possibility if (5) is to hold provided we subject the star to the following physical conditions:

(1) T is finite at $m = 0$ *i.e.* at the centre

(2) $T = 0$ at the surface where m is finite.

The proof runs as follows.

Case I. $x = 0$

In this case (5) gives

$$\frac{m}{\lambda} = \int_0^m T^y \cdot dm$$

which on differentiation gives

$$\frac{1}{\lambda} = T^y$$

which leads to either

$$T = const. \text{ (non zero)}$$

or

$$y = 0, \ \lambda = 1.$$

$T = const.$ and n is to be finite leads to, by virtue of (3) and (4)

$$P = const.$$

which is absurd and thus the case

$$T = const.$$

is ruled out. The second possibility, however, is admitted. $T = const.$ (non-zero) also contradicts physical condition (2).

Case II. $x \neq 0$, $x > y$ (the inequality holding strictly)

Now, (5) gives

$$\frac{d(T^x m)}{(T^x m)^{y/x}} = \lambda \cdot \frac{dm}{m^{y/x}}$$

which on integration gives

$$(T^x m)^{1-y/x} = \lambda \cdot m^{1-y/x} + const.$$

But since T is finite at $m = 0$ it follows from this equation that the constant of integration must vanish and we get

$$T^{x-y} = \lambda$$

which gives $x = y$, ($T = const.$ being ruled out for reason stated in Case I) thereby leading to contradiction of the hypothesis.

Case III $x \neq 0$, $x = y$.

In this case we get from (5)

$$m = (const. \times T^{x}m)^{1/\lambda} \tag{7}$$

which gives

$$m \propto T^{x/(\lambda-1)} \qquad \text{provided } \lambda \neq 1$$

which can never yield the result that T is finite when $m = 0$.

If, however

$$\lambda = 1$$

we have, from (7)

$$T^{x} = const.$$

i.e.

$$T = const. \text{ (since } x \neq 0).$$

This alternative is ruled out for reason stated in Case I.

Case III. $x \neq 0$, $x < y$

As in case II we get

$$(T^{x}m)^{1-y/x} = \lambda \cdot m^{1-y/x} + const.$$

or

$$1 = \lambda \cdot T^{y-x} + const \cdot T^{y-x}(m)^{y/x-1}.$$

This cannot give $T = 0$ for any finite m.

Thus we see that

$$x = y = 0, \qquad \lambda = 1 \tag{8}$$

is the only possibility when the star is a polytrope. Further it follows obviously that when (8) is satisfied the star is a polytrope, the constant A and the polytropic index, n, being given by the relations

$$\lambda = 1$$

and

$$x = 0$$

respectively. Elimination of n between $x = 0$ and $y = 0$ will lead to the requisite relation between α, ν, b, w. Also $y = 0$ implies uniformity of subatomic energy generation, (physically or effectively).

Conclusion. Eqn. (8) with the help of eqns. (6) gives us

$$3-\nu-(1+\alpha)n = 0.$$

$$w+nb = 0,$$

as conditions for the star to be a polytrope. The following cases are possible:

Case I.

$$w = 0,\ b = 0,\ n = \frac{3-\nu}{1+\alpha}$$

i.e. a star with uniform rate of energy generation throughout can be a polytrope.

Case II. $w = 0,\ n = 0,\ \nu = 3.$

A homogeneous star is possible with generation independent of temperature and opacity varying as the cube of temperature This case has to be ruled out on physical grounds (though it does not infringe the mathematical conditions taken as basis in this paper).

Case III. $$(3-\nu)b + (1+\alpha)w = 0. \tag{9}$$

In this Case

$$n = \frac{3-\nu}{1+\alpha} = -\frac{w}{b}.$$

Hence a polytropic star is possible when the exponents occuring in (1) satisfy (9).

This case has also to be ruled out as neither w nor b is negative in any physical law. However the interesting point to note is that in this (hypothetical) case though the physical law does not make the rate of generation uniform effectively it becomes so by virtue of the relation

$$w + nb = 0.$$

Acknowledgement

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In Memorium

PROFESSOR CUTHBERT EDMUND CULLIS

The members of the Calcutta Mathematical Society received (April, 1954), with great sorrow the news of the death of Professor C. E. Cullis Foundation Member of the Society. With him passed away the last of the band of great enthusiasts who founded the Calcutta Mathematical Society in 1908 and steered it safe through the trying period of its early days. Himself, a great scholar with established reputation as a teacher of Mathematics, and with zeal for study and research in Mathematics, Professor Cullis became the righthand man of the late Sir Asutosh Mookerjee in his great work of reconstruction of the University, specially with regard to mathematical studies. Those were indeed Homeric days for Calcutta as a seat of learning. The old Examining body was gradually shaped by the sole efforts of Sir Asutosh into a teaching University for Post Graduate studies; the Hardinge Professorship in Pure Mathematics was founded and with the munificent gift from the late Sir Rashbehary Ghose, a Chair for Applied Mathematics was also created. Prof. Cullis was later called to be the Hardinge Professor of Pure Mathematics and take charge of this department of Post-Graduate studies. He occupied this chair till his retirement in 1922.

The activities of Professor Cullis were confined to Calcutta and to the Bengal Engineering College in its suburb. In the Presidency College, and in the University of Calcutta he had taught Mathematics to nearly one generation of students, many of whom later distinguished themselves in India and abroad. It is interesting to note that both S. N. Bose and M. N. Saha studied Mathematics with him in the Post-Graduate Classes in the Presidency College, Calcutta. Students of Mathematics considered it a great privilege to be a pupil of Prof. Cullis. In dealing with a subject he always started from first principles, which he was never tired of stressing, and developed the most intricate points in his characteristic lucid style. As he had a low voice he used to write out all arguments in full on the blackboard—a stupendous task by itself. Prof. Cullis was greatly loved and revered by his students. The writer happens to be one of the pupils of Prof. Cullis and still remembers vividly how all his fellow classmates were united in their sincere love and admiration for their distinguished teacher.

C. E. Cullis was born on the 15th of April, 1868 and was educated at King Edward VI High School for boys, Birmingham, England. He got prizes in Classics and Modern Languages and was a good athlete. He finally chose Mathematics as his special subject for study, got a scholarship from his school and entered the Caius College, Cambridge. For reasons of health he had for a year to take to travel and went to Australia and New Zealand in sailing ships. After taking the two parts of the Tripos Examination he gained the Smith's prize in Mathematics for a work entitled "The movements of perforated solids in incompressible fluids". For a year he studied in the German Universities of Jena and Berlin, where he got his Ph.D. degree.

C. E. Cullis came out to India, first as Professor of Mathematics in the Bengal Engineering College, Sibpur, near Calcutta, and was later transferred to the Presidency College, Calcutta. Here he remained for a number of years till he was called to the Hardinge Chair for Pure Mathematics. In the Presidency College a large number of youngmen who came to study mathematics attended his lectures. Some of them kept correspondence with him even for a long time after his retirement. Professor Cullis enjoyed his teaching and his students were devoted to him. After retirement from the University of Calcutta, Prof. Cullis settled in Gloucester and carried on his mathematical studies even up to a very old age. At Christmas 1953, he suffered from gastric ulcer and had to undergo an operation At first he seemed to recover, but a sudden collapse occurred on March 20, 1954. He is survived by a sister who communicated the sorrowful news to the Society.

Prof. Cullis took great interest in the Calcutta Mathematical Society, of which he was a Life member, even to the last days of his life. Being aware of the difficulties of publications in India he has left a legacy of £500 to the Society to be used for meeting its publication expenses. The Society received this sum gratefully as the last gift from one of its great benefactors.

N. R. Sen.

CERTAIN THEOREMS ON SELF-RECIPROCAL RELATIONSHIP IN OPERATIONAL CALCULUS—I

BY

B. N. BOSE, *Calcutta.*

(*Received—March 23, 1953*)

If $\varphi(p)$ be the Laplace Transform of a function $f(t)$, then

$$\varphi(p) = p\int_0^\infty e^{-pt}f(t)\,dt, \tag{i}$$

where p is a positive number (or a number whose real part is positive). We usually say that $\varphi(p)$ is operationally related to $f(t)$ and that $f(t)$ is the interpretation of $\varphi(p)$. Following McLachlan and Humbert, we write sombolically.

$$\varphi(p)\subset f(t), \text{ or } f(t)\supset\varphi(p).$$

We have recently obtained certain theorems (Bose, 1952 and Mitra and Bose, 1953) bearing on self-reciprocal relationship between $\varphi(x)$ and $f(x)$. The object of the present note is to obtain a generalisation of some of them.

Following Hardy and Titchmarsh (1937) we shall say that if a function be self-reciprocal in the Hankel Transform of order ν, then it is R_ν and further that a function is R_c or R_s according as it is $R_{-\frac{1}{2}}$, or $R_{\frac{1}{2}}$.

The following results of the Operational Calculus will be required for our purpose:

$$t^n f(t)\supset(-1)^n p\,\frac{d^n}{dp^n}\left[\frac{\varphi(p)}{p}\right] \tag{ii}$$

$$\left(t\frac{d}{dt}\right)^n f(t)\supset(-1)^n\left(p\frac{d}{dp}\right)^n\varphi(p) \tag{iii}$$

$$f^n(t)\supset p^n\varphi(p), \text{ if } f^r(0)=0,\ r=0, 1, 2, \ldots, n-1 \tag{iv}$$

and Goldstein's theorem giving (Goldstein, 1932)

$$\int_0^\infty\frac{\varphi(t)g(t)dt}{t}=\int_0^\infty\frac{f(t)\psi(t)dt}{t}, \tag{v}$$

where $\varphi(p)\subset f(t)$ and $\psi(p)\subset g(t)$, provided the integrals converge and the changes in the order of integrations required are justifiable.

We shall also make use of the well-known theorem due to Lerch (1903), which states that if $f(t)$ be a continuous function satisfying it.

We know that

$$\frac{p}{(p^2+b^2)^{\nu+\frac{1}{2}}}\subset\frac{\pi^{\frac{1}{2}}}{2^\nu\Gamma(\nu+\frac{1}{2})}\left(\frac{t}{b}\right)^\nu J_\nu(bt),\ \text{R}(\nu)>-\tfrac{1}{2} \tag{vi}$$

and also on writing $\nu+1$ for ν, we have

$$\frac{p}{(p^2+b^2)^{\nu+3/2}} \subset \frac{\pi^{\frac{1}{2}}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}\left(\frac{t}{b}\right)^{\nu+1} J_{\nu+1}(bt), \quad \mathrm{R}(\nu) > -\tfrac{3}{2}. \qquad \text{(vi a)}$$

Taking the relations (ii) and (vi a) and applying Goldstein's theorem, we get

$$\int_0^\infty \frac{b^2 x^n f(x)}{(b^2+x^2)^{\nu+3/2}}dx = \frac{(-1)^n \pi^{\frac{1}{2}} b^{-\nu+1}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} \int_0^\infty x^{\nu+1} J_{\nu+1}(bx) \frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]dx, \quad (R(\nu) > -1) \qquad (1)$$

the change in the order of integration involved in the process being justified, provided the following integrals converge:

(1) $$\int_0^1 |F(x)|\, dx, \text{ where } F(x) = x^n f(x)$$

and (2) $$\int_1^\infty x^{-N} | F(x) |\, dx. \text{ where } N = [\nu+\tfrac{1}{2}], \text{ or} \int_1^\infty x^{-(2\nu+3)} | F(x) |\, dx$$

according as N or $2\nu+3$ is smaller.

On putting $b = p$ in (1) and then intepreting with the help of the relation

$$\frac{p^2}{(p^2+b^2)^{\nu+3/2}} \subset \frac{\pi^{\frac{1}{2}}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} \frac{t^{\nu+1}}{b^\nu} J_\nu(bt), \quad R(\nu) > -1 \qquad \text{(vii)}$$

we get

$$t^{\nu+1}\int_0^\infty J_\nu(xt)x^{n-\nu}f(x)dx \supset (-1)^n p^{-\nu+1}\int_0^\infty J_{\nu+1}(px)x^{\nu+1}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]dx, \quad R(\nu) < -1$$

i.e.,

$$t^{\nu+\frac{1}{2}}\int_0^\infty (xt)^{\frac{1}{2}}J_\nu(xt)x^{n-\nu-\frac{1}{2}}f(x)dx \supset (-1)^n p^{-\nu+\frac{1}{2}}\int_0^\infty (px)^{\frac{1}{2}}J_{\nu+1}(px)x^{\nu+\frac{1}{2}}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]dx,$$

$$(R(\nu) > -1). \qquad (2)$$

We obtain the relation (2) by applying De La Vallée Poussin's theorem for changing the order of integration on the left hand side to the ranges $0 < \varepsilon \leqq x \leqq \infty$ and $0 < \eta \leqq t \leqq \infty$ and then making $\varepsilon \to 0$ and $\eta \to 0$ and the convergence of integrals required for this is that of (1) $\int_0^1 | F(x) |\, dx$ and (2) $\int_1^\infty x^{-\nu-\frac{1}{2}} | F(x) |\, dx$, where, as before, we write $F(x) = x^n f(x)$.

Now, let $x^{n-\nu-\frac{1}{2}}f(x)$ be R_ν. Then from (2) we have

$$t^n f(t) \supset (-1)^n p^{-\nu+\frac{1}{2}}\int_0^\infty (px)^{\frac{1}{2}}J_{\nu+1}(px)x^{\nu+\frac{1}{2}}\frac{d}{dx^2}\left[\frac{\varphi(x)}{x}\right]dx.$$

Hence by virtue of (ii), we get

$$(-1)^n p^{-\nu+\frac{1}{2}}\int_0^\infty (px)^{\frac{1}{2}}J_{\nu+1}(px)x^{\nu+\frac{1}{2}}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]dx = (-1)^n p\frac{d^n}{dp^n}\left[\frac{\varphi(p)}{p}\right]$$

i.e.,

$$\int_0^\infty (px)^{\frac{1}{2}}J_{\nu+1}(px)x^{\nu+\frac{1}{2}}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]dx = p^{\nu+\frac{1}{2}}\frac{d^n}{dp^n}\left[\frac{\varphi(p)}{p}\right], \quad R(\nu) > -1 \qquad (3)$$

which shows that $x^{\nu+\frac{1}{2}}\dfrac{d^n}{dx^n}\left[\dfrac{\varphi(x)}{x}\right]$ is $R_{\nu+1}$.

We thus have

Theorem A. *If* $x^{n-\nu-\frac{1}{2}}f(x)$ *be* R_ν, *then* $x^{\nu+\frac{1}{2}}\dfrac{d^n}{dx^n}\left[\dfrac{\varphi(x)}{x}\right]$ *is* $R_{\nu+1}$, *provided* $\int_0^\infty |f(x)|\,x^n\,dx$ *and* $\int_0^\infty x^{n-\nu-\frac{1}{2}}|f(x)|\,dx$ *converge.*

Again let $x^{\nu+\frac{1}{2}}\dfrac{d^n}{dx^n}\left[\dfrac{\varphi(x)}{x}\right]$ be $R_{\nu+1}$. Then from (2) we get

$$t^{\nu+\frac{1}{2}}\int_0^\infty (xt)^{\frac{1}{2}}J_\nu(xt)x^{n-\nu-\frac{1}{2}}f(x)dx \supset (-1)^n p\,\frac{d^n}{dp^n}\left[\frac{\varphi(p)}{p}\right].$$

Hence from (ii), provided $x^nf(x)$ is continuous in the arbitrary interval $(0, X)$, by Lerch's theorem we get

$$t^{\nu+1}\int_0^\infty (xt)^{\frac{1}{2}}J_\nu(xt)x^{n-\nu-\frac{1}{2}}f(x)dx = t^nf(t) \tag{4}$$

i.e.,

$$\int_0^\infty (xt)^{\frac{1}{2}}J_\nu(xt)x^{n-\nu-\frac{1}{2}}f(x)dx = t^{n-\nu-\frac{1}{2}}f(t),\quad R(\nu)>-1$$

showing that $x^{n-\nu-\frac{1}{2}}f(x)$ is R_ν.

We, therefore, have conversely,

Theorem B. *If* $x^{\nu+\frac{1}{2}}\dfrac{d^n}{dx^n}\left[\dfrac{\varphi(x)}{x}\right]$ *be* $R_{\nu+1}$ *and* $x^nf(x)$ *be continuous in* $(0, X)$, *then* $x^{n-\nu-\frac{1}{2}}f(x)$ *is* R_ν, *provided* $\int_0^\infty x^{n-\nu-\frac{1}{2}}|f(x)|\,dx$ *converges.*

Particular cases. Theorems I and II proved before elsewhere (Mitra and Bose, 1953) follow as particular cases of Theorems A and B respectively when $n = 0$.

Also Theorems V and VI provided before in the same paper become merely particular cases of Theorems A and B respectively when we put $n = 0$ and $\nu = -\frac{1}{2}$.

Again applying the formula (iv) when $n = 1$ to (ii), we get

$$\frac{d}{dt}[t^nf(t)] \supset (-1)^n p^2\,\frac{d^n}{dp^n}\left[\frac{\varphi(p)}{p}\right], \tag{viii}$$

provided $\underset{t\to 0}{Lt}[t^nf(t)] = 0.$

Now taking the relations (vii) and (viii) and applying Goldstein's theorem, we get

$$\int_0^\infty \frac{bx\,(d/dx)[x^nf(x)]\,dx}{(x^2+b^2)^{\nu+3/2}} = \frac{(-1)^n\pi^{\frac{1}{2}}b^{-\nu+1}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}\int_0^\infty x^{\nu+2}J_\nu(bx)\,\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]dx,\quad R(\nu)>-1 \tag{5}$$

provided, as in (1), the following integrals converge:

(1) $$\int_0^1 |F(x)|\,dx, \text{ where } F(x)=\frac{d}{dx}[x^n f(x)]$$

and (2) $\int_1^\infty x^{-N}\,|F(x)|\,dx,\ N=[\nu+\tfrac{1}{2}]$, or $\int_1^\infty x^{-(2\nu+2)}\,|F(x)|\,dx$, according as N or $2\nu+2$ is smaller.

Let us put $b=p$ in (5) and then interpret the result by means of the relation (vi *a*) and we get

$$t^{\nu+1}\int_0^\infty x^{-\nu}J_{\nu+1}(xt)\frac{d}{dx}[x^n f(x)]\,dx \supset (-1)^n p^{-\nu+1}\int_0^\infty x^{\nu+2}J_\nu(px)\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]dx,$$

i.e.,
$$t^{\nu+\frac{1}{2}}\int_0^\infty (xt)^{\frac{1}{2}}J_{\nu+1}(xt\cdot x^{-\nu-\frac{1}{2}}\frac{d}{dx}[x^n f(x)]\,dx \supset (-1)^n p^{-\nu+\frac{1}{2}}\int_0^\infty (px)^{\frac{1}{2}}J_\nu(px)\,x^{\nu+3/2}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]dx,$$
$$R(\nu)>-1 \qquad (6)$$

provided the integrals (obtained in a way similar to that for (2)), (1) $\int_0^1 x\,|F(x)|\,dx$, where $F(x)=\frac{d}{dx}[x^n f(x)]$ and (2) $\int_1^\infty x^{-\nu-\frac{1}{2}}\,|F(x)|\,dx$ converge.

Now, let $x^{-\nu-\frac{1}{2}}\frac{d}{dx}[x^n f(x)]$ be $R_{\nu+1}$; then from (6) we have

$$\frac{d}{dx}[t^n f(t)] \supset (-1)^n p^{-\nu+\frac{1}{2}}\int_0^\infty (px)^{\frac{1}{2}}J_\nu(px)\,x^{\nu+3/2}\frac{d^n}{dp^n}\left[\frac{\varphi(p)}{p}\right],\ R(\nu)>-1 \qquad (7)$$

showing that $x^{\nu+3/2}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]$ is R_ν.

We thus have

Theorem C. *If* $x^{-\nu-\frac{1}{2}}\frac{d}{dx}[x^n f(x)]$ *be* $R_{\nu+1}$ *and* $\underset{x\to 0}{Lt}[x^n f(x)]=0$, *then* $x^{\nu+3/2}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]$ *is* R_ν, *provided the convergence of the integrals referred to in (6) holds good.*

Conversely, let $x^{\nu+3/2}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]$ *be* R_ν; *then from (6) we get*

$$t^{\nu+\frac{1}{2}}\int_0^\infty (xt)^{\frac{1}{2}}J_{\nu+1}(xt)x^{-\nu-\frac{1}{2}}\frac{d}{dx}[x^n f(x)]\,dx \supset (-1)^n p^{2}\frac{d^n}{dp^n}\left[\frac{\varphi(p)}{p}\right].$$

Therfore from (viii); provided $\frac{d}{dx}[x^n f(x)]$ is continuous in (0, X), by Lerch's theorem we get

$$t^{\nu+\frac{1}{2}}\int_0^\infty (xt)^{\frac{1}{2}}J_{\nu+1}(xt)x^{-\nu-\frac{1}{2}}\frac{d}{dx}[x^n f(x)]\,dx = \frac{d}{dt}[t^n f(t)];$$

i.e.,
$$\int_0^\infty (xt)^{\frac{1}{2}} J_{\nu+1}(xt)\, x^{-\nu-\frac{1}{2}} \frac{d}{dx}[x^n f(x)]\,dx = t^{-\nu-\frac{1}{2}} \frac{d}{dt}[t^n f(t)], \quad R(\nu) > -1 \quad \ldots \ldots$$

proving that $x^{-\nu-\frac{1}{2}} \frac{d}{dx}[x^n f(x)]$ is $R_{\nu+1}$.

Thus we have

Theorem D. *If* $x^{\nu+3/2} \frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]$ *be* R_ν *and* $\frac{d}{dx}[x^n f(x)]$ *is continuous in* $(0, X)$ *and* $\underset{x\to 0}{Lt}[x^n f(x)] = 0$, *then* $x^{-\nu-\frac{1}{2}} \frac{d}{dx}[x^n f(x)]$ *is* $R_{\nu+1}$, *provided the convergence of the integral* $\int_1^\infty x^{-\nu-\frac{1}{2}} \left| \frac{d}{dx}[x^n f(x)] \right| dx$ *holds good.*

Particular Cases. Theorems III and IV provod in the same paper referred to above follow as particular cases of Theorems D and C respectively if we put $n = 0$ and $\nu = -\frac{1}{2}$.

Let us next take (vi *a*) and (viii) and apply Goldstein's theorem. We then have

$$\int_0^\infty \frac{b^{\frac{1}{2}}(d/dx)[x^n f(x)]\,dx}{(b_1 + x^2)^{\nu+3/2}} = \frac{(-1)^n \pi^{\frac{1}{2}} b^{-\nu+1}}{2^{\nu+1}\Gamma(\nu+\frac{1}{2})} \int_0^\infty x^{\nu+1} J_{\nu+1}(bx) \frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right] dx, \quad R(\nu) > -1 \qquad (9)$$

provided the convergence of integrals referred to in (1) with $\psi(x) = (d/dx)[x^n f(x)]$ holds good.

Further putting $b = p$ in (9) and then interpreting the result by means of (vii), we get

$$t^{\nu+\frac{1}{2}} \int_0^\infty (xt)^{\frac{1}{2}} J_\nu(xt)\, x^{-\nu-\frac{1}{2}} \frac{d}{dx}[x^n f(x)]\,dx \supset (-1)^n p^{-\nu+\frac{1}{2}} \int_0^\infty (px)^{\frac{1}{2}} J_{\nu+1}(px)\, x^{\nu+2/3} \frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right] dx,$$
$$R(\nu) > -1 \qquad (10)$$

provided the convergence of integrals referred to (for the relation (2)) with $F(x) = d/dx[x^n f(x)]$ holds good.

Then proceeding as in Theorems C and D, we easily get the following:

Theorem E. *If* $x^{-\nu-\frac{1}{2}} \frac{d}{dx}[x^n f(x)]$ *be* R_ν *and* $\underset{x\to 0}{Lt}[x^n f(x)] = 0$, *then* $x^{\nu+3/2} \frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]$ *is* $R_{\nu+1}$, *provided the convergence of integrals referred to in (10) viz.*, $\int_0^1 |F(x)|\,dx$ *and* $\int_1^\infty x^{-\nu-\frac{1}{2}} |F(x)|\,dx$, *where* $F(x) = \frac{d}{dx}[x^n f(x)]$, *holds good.*

Theorem F. *Conversely, if* $x^{\nu+3/2}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]$ *be* $R_{\nu+1}$ *and* $\frac{d}{dx}[x^n f(x)]$ *is continuous in* $(0, X)$ *and* $\underset{x\to 0}{Lt}[x^n f(x)] = 0$, *then* $x^{-\nu-\frac{1}{2}}\frac{d}{dx}[x^n f(x)]$ *is* R_ν, *provided the integral* $\int_1^\infty x^{-\nu-\frac{1}{2}}\left|\frac{d}{dx}[x^n f(x)]\right| dx$ *is convergent.*

Particular Cases. On putting $n = 0$ and $\nu = -\frac{1}{2}$, we get Theorems I and II proved before elswhere (Bose, 1952), as particular cases of Theorems F and E respectively.

Further applying Goldstein's theorem to the relations (ii) and (vii), we get

$$\int_0^\infty \frac{bx^{n+1}f(x)dx}{(b^2+x^2)^{\nu+3/2}} = \frac{\pi^{\frac{1}{2}}(-1)^n b^{-\nu+1}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}\int_0^\infty x^{\nu+1}J_\nu(bx)\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right],\quad R(\nu)>-1 \tag{11}$$

provided the conditions regarding convergence of integrals referred to in (5) with $F(x) = x^n f(x)$ hold good.

Then putting $b = p$ in (11) and interpreting the result we get

$$t^{\nu+\frac{1}{2}}\int_0^\infty (xt)^{\frac{1}{2}}J_\nu(xt)x^{n-\nu-\frac{1}{2}}f(x)dx \supset (-1)^n p^{-\nu+\frac{1}{2}}\int_0^\infty (px)^{\frac{1}{2}}J_\nu(px)x^{\nu+\frac{1}{2}}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]dx,\quad R(\nu)<-1 \tag{12}$$

provided the integrals referred to in (6) with $F(x) = x^n f(x$ converge.

Then we easily get the following results:

Theorem G. *If* $x^{n-\nu-\frac{1}{2}}f(x)$ *be* $R_{\nu+1}$, *then* $x^{\nu+\frac{1}{2}}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]$ *is* R_ν *provided* $\int_0^1 x^{n+1}|f(x)|dx$ *and* $\int_1^\infty x^{n-\nu-\frac{1}{2}}|f(x)|dx$ *converge.*

Theorem H. *Conversely, if* $x^{\nu+\frac{1}{2}}\frac{d^n}{dx^n}\left[\frac{\varphi(x)}{x}\right]$ *be* R_ν *and* $x^n f(x)$ *is continuous in* $(0, X)$ *then* $x^{n-\nu-\frac{1}{2}}f(x)$ *is* $R_{\nu+1}$, *provided* $\int_1^\infty x^{n-\nu-\frac{1}{2}}|f(x)|dx$ *converges.*

Particular cases. If we put $n = 0$ and $\nu = -\frac{1}{2}$, we arrive at Theorems III and IV proved before (Bose, 1952), as particular cases of Theorems H and G respectively.

Again, applying Goldstein's theorem to the relations (iii) and (vii), get

$$\int_0^\infty \frac{bx(x(d/dx)^n)f(x)dx}{(x^2+b^2)^{\nu+3/2}} = \frac{(-1)^n\pi^{\frac{1}{2}}b^{-\nu+1}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}\int_0^\infty x^\nu J_\nu(bx)\left(x\frac{d}{dx}\right)^n\varphi(x)\,dx,\quad R(\nu)>-1 \tag{13}$$

provided the integrals referred to in (5) with $F(x) = (x(d/dx))^n\ F(x)$ converge.

Further putting $b = p$ in (13) and interpreting the result by means of (vi a), we get

$$t^{\nu+\frac{1}{2}}\int_0^\infty (xt)^{\frac{1}{2}}J_{\nu+1}(xt)x^{-\nu-\frac{1}{2}}\left(x\frac{d}{dx}\right)^n f(x)dx \supset (-1)^n p^{-\nu+\frac{1}{2}}\int_0^\infty (px)^{\frac{1}{2}}J_\nu(px)x^{\nu-\frac{1}{2}}\left(x\frac{d}{dx}\right)^n \varphi(x)\, dx,$$

$$R(\nu) > -1 \qquad (14)$$

provided the integrals mentioned in (6) with $F(x) = (x(d/dx))^n f(x)$ converge.

We then easily deduce the following results:

Theorem I. *If* $x^{-\nu-\frac{1}{2}}\left(x\frac{d}{dx}\right)^n f(x)$ *be* $R_{\nu+1}$, *then* $x^{\nu-\frac{1}{2}}\left(x\frac{d}{dx}\right)^n \varphi(x)$ *is* R_ν, *provided the convergence of the integrals referred to in (14) holds good.*

Theorem J. *Conversely, if* $x^{\nu-\frac{1}{2}}\left(x\frac{d}{dx}\right)^n \varphi(x)$ *be* R_ν *and* $\left(x\frac{d}{dx}\right)^n f(x)$ *be continuous in* $(0, X)$ *then* $x^{-\nu-\frac{1}{2}}\left(x\frac{d}{dx}\right)^n f(x)$ *is* $R_{\nu+1}$, *provided* $\int_1^\infty x^{-\nu-\frac{1}{2}}\left|\left(x\frac{d}{dx}\right)^n f(x)\right| dx$ *is convergent.*

Particular cases. When $n = 1$, Theorems I and J lead to Theorems XIII and XIV respectively (Bose, 1952 [2]).

Treating finally the relations (iv) and (vii) as before we first get

$$\int_0^\infty \frac{bxf^n(x)dx}{(b^2+x^2)^{\nu+3/2}} = \frac{\pi^{\frac{1}{2}}b^{-\nu+1}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}\int_0^\infty x^{n+\nu}J_\nu(bx)\varphi(x)dx, \quad R(\nu) > -1 \qquad (15)$$

and then

$$t^{\nu+\frac{1}{2}}\int_0^\infty (xt)^{\frac{1}{2}}J_{\nu+1}(xt)x^{-\nu-\frac{1}{2}}f(x)dx \supset p^{-\nu+\frac{1}{2}}\int_0^\infty (px)^{\frac{1}{2}}J_\nu(px)x^{n+\nu-\frac{1}{2}}\varphi(x)dx, \quad R(\nu) > -1 \quad (16)$$

provided the integrals referred to in (6) but with $F(x) = f^n(x)$ convege.

Hence we deduce the following results:

Theorem K. *If* $x^{-\nu-\frac{1}{2}}f^n(x)$ *be* $R_{\nu+1}$ *and* $f^r(0) = 0$, $r = 0, 1, \ldots, n-1$, *then* $x^{n+\nu-\frac{1}{2}}\varphi(x)$ *is* R_ν, *provided* $\int_1^\infty x\,|f^n(x)|\,dx$ *and* $\int_0^\infty x^{-\nu-\frac{1}{2}}|f^n(x)|\,dx$ *converge.*

Theorem L. *Conversely, if* $x^{n+\nu-\frac{1}{2}}\varphi(x)$ *be* R_ν *and further* $f^r(0)=0$, $r=0, 1, \ldots, n-1$ *and* $f^n(x)$ *is continuous in* $(0, X)$ *then* $x^{-\nu-\frac{1}{2}}f^n(x)$ *is* $R_{\nu+1}$, *provided* $\int_1^\infty x^{-\nu-\frac{1}{2}}\,|f^n(x)|\,dx$ *converges.*

Theorems XV and XVI (Bose, 1952) are but particular cases of Theorems K and L respectively when $n = 1$.

Examples. We now proceed to give certain examples of some of the theorems just established.

The examples we construct involve MacRobert's E-function (1942), which is denoted by the symbol $E(\alpha, \beta :: x)$ and is defined as

$$E(\alpha, \beta :: x) = \Gamma(\alpha) \int_0^\infty e^{-\lambda}\lambda^{\beta-1}\left(1+\frac{\lambda}{x}\right)^{-\alpha} d\lambda; \; R'(\beta)>0 \tag{a}$$

which is equal to $\Gamma(\alpha)\Gamma(\beta)e^{\frac{1}{2}x}x^{\frac{1}{2}(\alpha+\beta-1)}W_{\frac{1}{2}-\frac{1}{2}(\alpha+\beta), \frac{1}{2}(\beta-\alpha)}$.

Further when $x \to \infty$, $E(\alpha, \beta :: x) \to \Gamma(\alpha)\Gamma(\beta)$ and when $x \to 0$, $E(\alpha, \beta :: x) \to 0$, provided $R(\alpha) > 0$ and $R(\beta) > 0$.

We shall make use of the following properties of the function proved by N. N. Bose (1950 and 1952) :

(1) the recurrence relations:

$$\beta E(\alpha, \beta :: x) - E(\alpha, \beta + 1 :: x) = (1/x)E(\alpha+1, \beta+1 :: x) \tag{b}$$

$$\alpha E(\alpha, \beta :: x) - E(\alpha+1, \beta :: x) = (1/x)E(\alpha+1, \beta+1 :: x) \tag{c}$$

(2) the Operational relations:

$$E(\alpha, \beta :: \tfrac{1}{2}p^2) \subset \Gamma(\alpha)\Gamma(\beta)\,{}_2F_2(\alpha, \beta; 1, \tfrac{1}{2}; -\tfrac{1}{2}t^2) \tag{d}$$

$$E(n+\tfrac{1}{2}, 1 :: \tfrac{1}{2}p^2) \subset \pi^{\frac{1}{2}}(-1)^n 2^{-n} \frac{d^{2n}}{dt^{2n}}(e^{-\frac{1}{2}t^2}) \tag{e}$$

(3) the self-reciprocal relation:

$$\int_0^\infty (px)^{\frac{1}{2}} J_\nu(px) x^{-\alpha-\frac{1}{2}} E\left(\alpha, \frac{\nu+\alpha+1}{2} :: \tfrac{1}{2}x^2\right) dx = p^{-\alpha-\frac{1}{2}} E\left(\alpha, \frac{\nu+\alpha+1}{2} :: \tfrac{1}{2}t^2\right), \; (\nu > -1) \tag{f}$$

which means that $p^{-\alpha-\frac{1}{2}}E\left(\alpha, \frac{\nu+\alpha+1}{2} :: \frac{1}{2}p^2\right)$ is R_ν.

On differentiating (*a*) which is assumed permissible, we get

$$\frac{d}{dx}E(\alpha, \beta :: x) = \frac{1}{x^2}E(\alpha+1, \beta+1 :: x),$$

whence, on writing $\frac{1}{2}x^2$ for x, we have

$$\frac{d}{dx}E(\alpha, \beta :: \tfrac{1}{2}x^2) = \frac{4}{x^3}E(\alpha+1, \beta+1 :: \tfrac{1}{2}x^2) \tag{g}$$

Also we have

$$\frac{d}{dx}\left[\frac{1}{x}E(\alpha, \beta :: \tfrac{1}{2}x^2)\right] = \frac{4}{x^4}E(\alpha+1, \beta+1 :: \tfrac{1}{2}x^2) - \frac{1}{x^2}E(\alpha, \beta :: \tfrac{1}{2}x^2)$$

and therefore by means of the recurrence relations (*b*) and (*c*), we get

$$\frac{d}{dx}\left[\frac{1}{x}E(\alpha, \beta :: \tfrac{1}{2}x^2)\right]$$

$$= \frac{(2\alpha-1)}{x^2}E(\alpha, \beta :: \tfrac{1}{2}x^2) - \frac{2}{x^2}E(\alpha+1, \beta :: \tfrac{1}{2}x^2) \tag{h}$$

$$= \frac{(2\beta-1)}{x^2}E(\alpha, \beta :: \tfrac{1}{2}x^2) - \frac{2}{x^2}E(\alpha, \beta+1 :: \tfrac{1}{2}x^2). \tag{i}$$

From (d) and (e), we easily get the operational relations:

$$E(1, \tfrac{1}{2} :: \tfrac{1}{2}p^2) \subset \pi^{\frac{1}{2}} e^{-\frac{1}{2}t^2} \qquad (j)$$

$$E(\tfrac{1}{2}, 1 :: \tfrac{1}{2}p^2) \subset \pi^{\frac{1}{2}} e^{-\frac{1}{2}t^2} \qquad (k)$$

$$E(\tfrac{3}{2}, 1 :: \tfrac{1}{2}p^2) \subset \tfrac{1}{2}\pi^{\frac{1}{2}}(1-t^2) e^{-\frac{1}{2}t^2} \qquad (l)$$

Also by writing $n+1$ for n in (e) and then combining the result with (e) by means of (b), we get

$$\frac{2}{p^2}E(n+\tfrac{3}{2}, 2 :: \tfrac{1}{2}p^2) \subset \frac{(-1)^n \pi^{\frac{1}{2}}}{2^{n+1}}\left[(2n+1)\frac{d^{2n}}{dt^{2n}}(e^{-\frac{1}{2}t^2}) + \frac{d^{2n+2}}{dt^{2n+2}}(e^{-\frac{1}{2}t^2})\right], \; n = 0, 1, 2, \ldots \qquad (m)$$

and in particular

$$\frac{2}{p^2}E(\tfrac{5}{2}, 2 :: \tfrac{1}{2}p^2) \subset \frac{\pi^{\frac{1}{2}}}{4}\frac{d}{dt}(-t^3 e^{-\frac{1}{2}t^2}). \qquad (n)$$

Further proceeding as in (h) we have

$$\frac{d}{dx}\left[\frac{1}{x^3}E(\alpha, \beta :: \tfrac{1}{2}x^2)\right]$$

$$= \frac{(2\alpha-3)}{x^4}E(\alpha, \beta :: \tfrac{1}{2}x^2) - \frac{2}{x^4}E(\alpha+1, \beta :: \tfrac{1}{2}x^2). \qquad (o)$$

Theorems A and B. Let us take $n = 1$ and $\nu = 0$. In this case Theorem B states that if $x^{\frac{1}{2}}\frac{d}{dx}\left[\frac{\varphi(x)}{x}\right]$ is R_1, then $x^{\frac{1}{2}}f(x)$ is R_0, provided $xf(x)$ is continuous.

Let $\varphi(p) = E(1, \tfrac{1}{2} :: \tfrac{1}{2}p^2)$, then by (j) $f(t) = \pi^{\frac{1}{2}}e^{-\frac{1}{2}t^2}$. Obviously $xf(x)$ is continuous.

By (i) we have

$$\frac{d}{dx}\left[\frac{\varphi(x)}{x}\right] = -\frac{2}{x^{2/8}}E(1, \tfrac{3}{2} :: \tfrac{1}{2}x^2)$$

which. by virtue of (f), is R_1.

We then expect that $x^{\frac{1}{2}}f(x)$, *i.e.*, $\pi^{\frac{1}{2}}x^{\frac{1}{2}}e^{-\frac{1}{2}x^2}$ is R_0.

It is known that the function (Mehrotra, 1934)

$$x^{\frac{1}{2}+\nu}e^{-\frac{1}{2}x^2} \qquad (p)$$

is R_ν. Hence $\pi^{\frac{1}{2}}x^{\frac{1}{2}}e^{-\frac{1}{2}x^2}$ is R_0.

We have thus an example of Theorem B. This is evidently an example of Theorem A as well.

Consider next $n = 0$ and $\nu = \frac{1}{2}$ and the theorems state that if $\varphi(x)$ be $R_{3/2}$, then $\frac{f(x)}{x}$ is $R_{\frac{1}{2}}$ and conversely.

Taking $\varphi(p) = \frac{2}{p^2}E(\tfrac{3}{2}, 2 :: \tfrac{1}{2}p^2)$, so that by (m) $f(t) = \frac{1}{2}\pi^{\frac{1}{2}}t^2 e^{-\frac{1}{2}t^2}$

we find that $\varphi(x)$ is $R_{3/2}$ by virtue of (f) and $\frac{f(x)}{x}$ *i.e.*, $\frac{1}{2}\pi^{\frac{1}{2}}xe^{-\frac{1}{4}x^2}$ is R_s.

Illustrations of the theorems when $n = 0$ and $\nu = -\frac{1}{2}$ are the following:

We take $\varphi(p) = E(\frac{1}{2}, 1 :: \frac{1}{2}p^2)$, then $f(t) = \pi^{\frac{1}{2}}e^{-\frac{1}{4}t^2}$ and we fiind that $\varphi(x)/x$ is R_s and $f(x)$ is R_0. This is what we should have according to the theorems.

Again applying Goldstein's theorems to the relations (McLachlan and Humbert, 1941)

$$\tfrac{1}{2}\pi^{\frac{1}{2}}p^{3/2}[H_{\frac{1}{4}}(\tfrac{1}{2}p^2)] - Y_{\frac{1}{4}}(\tfrac{1}{2}p^2)] \subset t^{\frac{1}{2}}J_{-\frac{1}{4}}(\tfrac{1}{2}t^2)$$

and

$$\frac{ap}{p^2+a^2} \subset \sin at$$

we get

$$\frac{\pi^{\frac{1}{2}}}{2}\int_0^\infty x^{\frac{1}{2}}[H_{\frac{1}{4}}(\tfrac{1}{2}x^2) - Y_{\frac{1}{4}}(\tfrac{1}{2}x^2)] \sin ax\, dx = a\int_0^\infty \frac{x^{\frac{1}{2}}J_{-\frac{1}{4}}(\frac{1}{2}x^2)dx}{x^2+a^2}.$$

But

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{x^2+a^2}J_{-\frac{1}{4}}(\tfrac{1}{2}x^2)\,dx = \frac{\pi}{2^{\frac{1}{2}}}a^{-\frac{1}{2}}[H^{\frac{1}{4}}(\tfrac{1}{2}a^2) - Y_{\frac{1}{4}}(\tfrac{1}{2}a^2)].$$

Hence we get that

$$\frac{\pi^{\frac{1}{2}}}{2}x^{\frac{1}{2}}[H_{\frac{1}{4}}(\tfrac{1}{2}x^2) - Y_{\frac{1}{4}}(\tfrac{1}{2}x^2)] \text{ is } R_s.$$

Taking

$$\varphi(p) = \frac{\pi^{\frac{1}{2}}}{2}p^{3/2}[H_{\frac{1}{4}}(\tfrac{1}{2}p^2) - Y_{\frac{1}{4}}(\tfrac{1}{2}p^2)]$$

so that

$$f(t) = t^{\frac{1}{2}}J_{-\frac{1}{4}}(\tfrac{1}{2}t^2),$$

we arrive at the fact that $\varphi(x)/x$ is R_s and $f(x)$ is R_0.

Theorems C and D. Let us take $n = 0$ and $\nu = \frac{1}{2}$. Then Theorem D states that if $x\varphi(x)$ be R_s, then $(1/x)f'(x)$ is $R_{3/2}$, provided $\underset{x\to 0}{Lt}[f(x)] = 0$ and $f'(x)$ continuous.

From (n) we have

$$\frac{2}{p^3}E(\tfrac{5}{2}, 2 :: \tfrac{1}{2}p^2) \subset \frac{\pi^{\frac{1}{2}}}{4}\int_0^t \frac{d}{dt}(-t^3e^{-\frac{1}{4}t^2})\,dt = -\frac{\pi^{\frac{1}{2}}}{4}t^3e^{-\frac{1}{4}t^2}$$

and therefore

$$\frac{2}{p^4}E(\tfrac{5}{2}, 2 :: \tfrac{1}{2}p^2) \subset -\frac{\pi^{\frac{1}{2}}}{4}\int_0^t t^3e^{-\frac{1}{4}t^2}dt = \frac{\pi^{\frac{1}{2}}}{4}[(t^2+2)e^{-\frac{1}{4}t^2} - 2]. \qquad (q)$$

We take $\varphi(p) = \frac{2}{p^4}E(\frac{5}{2}, 2 :: \frac{1}{2}p^2)$, then we have $f(t) = -\frac{\pi^{\frac{1}{2}}}{4}[(t^2+2)e^{-\frac{1}{4}t^2} - 2]$.

Evidently $\underset{x\to 0}{Lt}[f(x)] = 0$ and $f\,(x) = \frac{\pi^{\frac{1}{2}}}{4}x^3e^{-\frac{1}{4}x^2}$ is continuous.

Now, $x\varphi(x) = (2/x^3)E(\frac{5}{2}, 2 :: \frac{1}{2}x^2)$, which, by (f), is R_s and $(1/x)f'(x) = (\pi^{\frac{1}{2}}/4)x^2e^{-\frac{1}{4}x^2}$ which, by (p), is $R_{3/2}$. Thus the above is an example of Theorem D and eveidently also of Theorem C, Let us again consider the case when $n = 0$ and $\nu = -\frac{1}{2}$. Treating the relation (l) as before, we get

$$\frac{1}{p} E(\tfrac{3}{2}, 1 :: \tfrac{1}{2}p^2) \subset \tfrac{1}{2}\pi^{\frac{1}{2}}\, t e^{-\frac{1}{2}t^2} \qquad (r)$$

and hence

$$\frac{1}{p} E(\tfrac{3}{2}, 1 :: \tfrac{1}{2}p^2) \subset \tfrac{1}{2}\pi^{\frac{1}{2}}[1 - e^{-\frac{1}{2}t^2}].$$

We take

$$\varphi(p) = \frac{1}{p^2} E(\tfrac{3}{2}, 1 :: \tfrac{1}{2}p^2)$$

so that

$$f(t) = \tfrac{1}{2}\pi^{\frac{1}{2}}(1 - e^{-\frac{1}{2}t^2}).$$

It is evident that $\underset{x \to 0}{Lt}\,[f(x)] = 0$ and $f(x) = \frac{1}{2}\pi^{\frac{1}{2}} x e^{-\frac{1}{2}x^2}$ is continuous.

We also find that $\varphi(x)$ is R_ν and $f'(x)$ is R_ν and this is what should have according to the theorems when $n = 0$ and $\nu = -\frac{1}{2}$.

Theorems E and F. Let us take $n = 1$ and $\nu = -\frac{1}{2}$ so that Theorem F gives that if $x\dfrac{d}{dx}\left[\dfrac{\varphi(x)}{x}\right]$ is R_ν, then $\dfrac{d}{dx}[xf(x)]$ is R_ν, provided $\dfrac{d}{dx}[xf(x)]$ is continuous and $\underset{x \to 0}{Lt}[xf(x)] = 0$.

From (o) we have

$$\frac{d}{dx}\left[\frac{1}{x^3} E(\tfrac{3}{2}, 2 :: \tfrac{1}{2}x^2)\right] = -\frac{2}{x^4} E(\tfrac{5}{2}, 2 :: \tfrac{1}{2}x^2)$$

Now let us take $\varphi(p) = \dfrac{1}{p^2} E(\tfrac{3}{2}, 2 :: \tfrac{1}{2}p^2)$, then by (m) $f(t) = \frac{1}{4}\pi^{\frac{1}{2}} t^2 e^{-\frac{1}{2}t^2}$.

The conditions regarding coutinuity and limit are easily found to be satisfied.

We have $x\dfrac{d}{dx}\left[\dfrac{\varphi(x)}{x}\right] = -2/x^3 E(\tfrac{5}{2}, 2 :: \tfrac{1}{2}x^2)$, which, by (f), is R_ν.

We therefore expect that $d/dx[xf(x)]$

i.e.,
$$\tfrac{1}{4}\pi^{\frac{1}{2}}[3x^2 - x^4]e^{-\frac{1}{2}x^2} \text{ is } R_\nu.$$

By applying Bailey's result (Bailey, 1930) that the function

$$g(x) = \left\{\sum_{r=0}^{n} \frac{(-n)_r x^{2r}}{2^r r!\,(1 + \frac{1}{2}\nu - \frac{1}{2}n)_r}\right\} x^{\frac{1}{2}+\nu} e^{-\frac{1}{2}x^2}$$

is $\pm R_\nu$ according as n even or odd integer and taking $n = 2$, we get that

$$\{1 + \tfrac{4}{3}(3x^2 - x^4)\} e^{-\frac{1}{2}x^2}$$

is R_ν. Hence, by virtue of the fact that $e^{-\frac{1}{2}x^2}$ is R_ν we finally get that

$$\tfrac{1}{4}\pi^{\frac{1}{2}}(3x^2 - x^4) e^{-\frac{1}{2}x^2}$$

is R_ν. The above is also an example of theorem E.

Theorems G and H. When $n = 1$ and $\nu = -\frac{1}{2}$, Theorem H states that if $\frac{d}{dx}\left[\frac{\varphi(x)}{x}\right]$ is R_0, then $xf(x)$ is $R_{\frac12}$, provided $xf(x)$ is continuous.

Let $\varphi(p) = E(\frac{1}{2}, 1 :: \frac{1}{2}p^2)$, then by (k) $f(t) = \pi^{\frac12}e^{-\frac14 t^2}$.

By (h) we have

$$\frac{d}{dx}\left[\frac{\varphi(x)}{x}\right] = \frac{2}{x^2}E(\tfrac{3}{2}, 1 :: \tfrac{1}{2}x^2),$$

which, by (f), is evidently R_0.

We then expect that $xf(x)$ i.e., $\pi^{\frac12}xe^{-\frac14 x^2}$ is $R_{\frac12}$, which is in fact so.

This serves as an illustration of theorem G also.

Again considering $n = 0$ and $\nu = -\frac{1}{2}$ and taking $\varphi(p) = \frac{1}{p}E(\frac{3}{2}, 1 :: \frac{1}{2}p^2)$, so that, by (r), $f(t) = \frac{1}{2}\pi^{\frac12}te^{-\frac14 t^2}$, we find that $\varphi(x)/x$ is R_0 and $f(x)$ is $R_{\frac12}$. This is what we should have by the theorems.

Theorems I and J. When $n = 1$ and $\nu = 0$, Theorem J states that if $x^{\frac12}\varphi'(x)$ is R_0, then $x^{\frac12}f'(x)$ is R_1, provided $xf'(x)$ is continuous.

We take $\varphi(p) = E(1, \frac{1}{2} :: p^2)$, then by (j) $f(t) = \pi^{\frac12}e^{-\frac14 t^2}$.

By (g) we have $x^{\frac12}\varphi'(x) = \frac{4}{x^{5/2}}E(2, \frac{3}{2} :: \frac{1}{2}x^2)$, which is evidently R_0 by (f)

Also $x^{\frac12}f'(x) = -\pi^{\frac12}x^{3/2}e^{-\frac14 x^2}$, which, by (p), is R_1 it is easy to see that $xf'(x)$ is continuous. Thus the above is an illustration of theorem J and also of Theorem I.

Theorems K and L. We find that the examples given for Theorems C and D both serve as illustrations of Theorems K and L in the cases when $n = 1$, $\nu = \frac{1}{2}$ and, $n = 1$, $\nu = -\frac{1}{2}$.

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STRESS DISTRIBUTION IN THIN AEOLOTROPIC PLATES—II

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1. Introduction. In a previous paper a method of Fourier transform of the stress function has been given for the solution of the generalised plane stress problems, when the material of the plate has two planes of elastic symmetry at right angles to each other and also at right angles to the plane of the plate. In the present paper, this method has been applied to find the stress distribution in (i) a semi-infinite plate with a concentrated force acting at an internal point and (ii) a strip of infinite length but of finite breadth acted on by (a) a concentrated force and (b) an isolated couple within it.

2. Expressions for the Stresses. It has been shown that the function $G(\eta, \xi)$ defined by

$$G(\eta, \xi) = \int_{-\infty}^{\infty} \chi(x, \eta) e^{i\xi x} dx \tag{1}$$

satisfies the differential equation

$$\left(\frac{d^4}{d\eta^4} - 2k\xi^2 \frac{d^2}{d\eta^2} + \xi^4\right) G(\eta, \xi) = 0 \tag{2}$$

where

$$\eta = \gamma y \tag{3}$$

γ and k being constants depending upon the elastic property of the material.

The stresses $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ can be expressed in terms of G as

$$\int_{-\infty}^{\infty} \widehat{xx} e^{i\xi x} dx = \frac{d^2G}{dy^2}\,; \quad \int_{-\infty}^{\infty} \widehat{yy} e^{i\xi x} dx = -\xi^2 G\,; \quad \int_{-\infty}^{\infty} \widehat{xy} e^{i\xi x} dx = i\xi \frac{dG}{dy} \tag{4}$$

or, in alternative forms, as

$$\widehat{xx} = \frac{1}{\pi} \int_0^{\infty} \frac{d^2G}{dy^2} \cos \xi x \, d\xi\,; \quad \widehat{yy} = -\frac{1}{\pi} \int_0^{\infty} \xi^2 G \cos \xi x \, d\xi, \quad \widehat{xy} = \frac{1}{\pi} \int_0^{\infty} \xi \frac{dG}{dy} \sin \xi x \, d\xi \tag{5}$$

when G is an even function in ξ, and

$$\widehat{xx} = -\frac{i}{\pi} \int_0^{\infty} \frac{d^2G}{dy^2} \sin \xi x \, d\xi\,; \quad \widehat{yy} = \frac{i}{\pi} \int_0^{\infty} \xi^2 G \sin \xi x \, d\xi\,; \quad \widehat{xy} = \frac{i}{\pi} \int_0^{\infty} \xi \frac{dG}{dy} \cos \xi x \, d\xi \tag{6}$$

when G is an odd function in ξ.

We also recall that, if a concentrated force F acts in the positive direction of the y-axis at the origin in an infinite plate, the stresses are given by

$$\left.\begin{aligned}
\widehat{xx} &= \frac{F\gamma^3}{2\pi(\alpha^4 - \beta^2)} \left[\frac{\sigma_1\alpha^2\gamma^2 + 1)\alpha y}{x^2 + \alpha^2\gamma^4 y^2} - \frac{(\sigma_1\beta^2\gamma^2 + 1)\beta y}{x_2 + \beta^2\gamma^4 y^2} \right] \\
\widehat{yy} &= -\frac{F\gamma y}{2\pi(\alpha^2 - \beta^2)} \left[\frac{(\sigma_1\alpha^2\gamma^2 + 1)\beta}{x^2 + \alpha^2\gamma^2 y^4} - \frac{(\sigma_1\beta^2\gamma^2 + 1)\sigma}{x^2 + \beta^2\gamma^2 y^4} \right] \\
\widehat{xy} &= -\frac{F\gamma x}{2\pi(\alpha^2 - \beta^2)} \left[\frac{(\sigma_1\alpha^2\gamma^2 + 1)\beta}{x^2 + \alpha^2\gamma^2 y^2} - \frac{(\sigma_1\beta^2\gamma^2 + 1)\alpha}{x^2 + \beta^2\gamma^2 y^2} \right]
\end{aligned}\right\} \tag{A}$$

while for a concentrated couple of moment M, the stresses are given by

$$\left.\begin{aligned}
\widehat{xx} &= -\frac{M\gamma^3 xy}{\pi(\alpha^2-\beta^2)}\left[\frac{\alpha(\sigma_1\alpha^2\gamma^2+1)}{(x^2+\alpha^2\gamma^2y^2)^2}-\frac{\beta(\sigma_1\beta^2\gamma^2+1)}{(x^2+\beta^2\gamma^2y^2)^2}\right.\\
\widehat{yy} &= \frac{M\gamma xy}{\pi(\alpha^2-\beta^2)}\left[\frac{\beta(\sigma_1\alpha^2\gamma^2+1)}{(x^2+\alpha^2\gamma^2y^2)^2}-\frac{\alpha(\sigma_1\beta^2\gamma^2+1)}{(x^2+\beta^2\gamma^2y^2)^2}\right.\\
\widehat{xy} &= \frac{M\gamma}{2\pi(\alpha^2-\beta^2)}\left[\frac{\beta(\sigma_1\alpha^2\gamma^2+1)(x^2-\alpha^2\gamma^2y^2)}{(x^2+\alpha^2\gamma^2y^2)^2}-\frac{\alpha(\sigma_1\beta^2\gamma^2+1)(x^2-\beta^2\gamma^2y^2)}{(x^2+\beta^2\gamma^2y^2)^2}\right]
\end{aligned}\right\} \quad (B)$$

where $\alpha = (k+(k^2-1)^{\frac{1}{2}})^{\frac{1}{2}}$; $\beta = (k-(k^2-1)^{\frac{1}{2}})^{\frac{1}{2}}$; $\alpha\beta = 1$.

3. **Concentrated force in a semi-infinite plate.** We consider a semi-infinite plate bounded by one straight edge $y = b$, this edge being parallel to one axis of symmetry (x-axis). If a concentrated force F acts along the y-axis at the origin in an infinite plate, the stresses $\widehat{xx}_1$, $\widehat{yy}_1$, $\widehat{xy}_1$ are given by the relations (A). On the edge $y = b$ these are

$$\left.\begin{aligned}
(\widehat{xx}_1)_{y=b} &= \frac{F\gamma^3 b}{2\pi(\alpha^2-\beta^2)}\left[\frac{(\sigma_1\alpha^2\gamma^2+1)\alpha}{x^2+\alpha^2\gamma^2b^2}-\frac{(\sigma_1\beta^2\gamma^2+1)\beta}{x^2+\beta^2\gamma^2b^2}\right].\\
(\widehat{yy}_1)_{y=b} &= -\frac{F\gamma b}{2\pi(\alpha^2-\beta^2)}\left[\frac{(\sigma_1\alpha^2\gamma^2+1)\beta}{x^2+\alpha^2\gamma^2b^2}-\frac{(\sigma_1\beta^2\gamma^2+1)\alpha}{x^2+\beta^2\gamma^2b^2}\right]\equiv-\varphi_1(x),\\
(\widehat{xy}_1)_{y=b} &= -\frac{F\gamma x}{2\pi(\alpha^2-\beta^2)}\left[\frac{(\sigma_1\alpha^2\gamma^2+1)\beta}{x^2+\alpha^2\gamma^4b^2}-\frac{(\sigma_1\beta^2\gamma^2+1)\alpha}{x^2+\beta^3\gamma^2b^2}\right]\equiv-\varphi_2(x).
\end{aligned}\right\} \quad (7)$$

To nullify the stresses $\widehat{yy}_1$ and $\widehat{xy}_1$ on the boundary, we are to superpose a stress system $\widehat{xx}_2$, $\widehat{yy}_2$, $\widehat{xy}_2$ such that they vanish as $y \to -\infty$ and

$$(\widehat{yy}_2)_{y=b} = \varphi_1(x), \quad (\widehat{xy}_2)_{y=b} = \varphi_2(x). \tag{8}$$

As a solution of (2), let us take

$$G = Ae^{|\xi|\alpha\gamma y}+Be^{|\xi|\beta\gamma y} \tag{9}$$

From (4) and (8), we have,

$$\left.\begin{aligned}
-\xi^2[Ae^{|\xi|\alpha\gamma b}+Be^{|\xi|\beta\gamma b}] &= 2\int_0^\infty \varphi_1(x)\cos\xi x\,dx \equiv \bar{\varphi}_1(\xi)\\
|\xi|\xi\gamma[\alpha Ae^{|\xi|\alpha\gamma b}+\beta Be^{|\xi|\beta\gamma b}] &= 2\int_0^\infty \varphi_2(x)\sin\xi x\,dx \equiv \bar{\varphi}_2(\xi).
\end{aligned}\right\} \quad (10)$$

These are satisfied if

$$A = \frac{\beta\bar{\varphi}_1/\xi^2+\bar{\varphi}_2/(|\xi|\xi\gamma)}{(\alpha-\beta)e^{|\xi|\alpha\gamma b}}, \quad B = -\frac{\alpha\bar{\varphi}_1/\xi^2+\bar{\varphi}_2/(|\xi|\xi}{(\alpha-\beta)e^{|\xi|\beta\gamma b}} \tag{11}$$

Since A and B are both even function in ξ, (5) gives,

$$\left.\begin{aligned}
\widehat{xx}_2 &= \frac{\gamma^2}{\pi}\int_0^\infty \xi^2(\alpha^2 A e^{\xi\alpha\gamma y} + \beta^2 B e^{\xi\beta\gamma y}) \cos \xi x \, d\xi. \\
\widehat{yy}_2 &= -\frac{1}{\pi}\int_0^\infty \xi^2(A e^{\xi\alpha\gamma y} + B e^{\xi\beta\gamma y}) \cos \xi x \, d\xi. \\
\widehat{xy}_2 &= \frac{\gamma}{\pi}\int_0^\infty \xi^2(\alpha A e^{\xi\alpha\gamma y} + \beta B e^{\xi\beta\gamma y}) \sin \xi x \, d\xi.
\end{aligned}\right\} \tag{12}$$

To evaluate $\bar{\varphi}_1(\xi)$ and $\bar{\varphi}_2(\xi)$, we substitute the expressions for $\varphi_1(x)$ and $\varphi_2(x)$ from (7) in $(10)_1$ and after integration, we have,

$$\bar{\varphi}_1(\xi) = \frac{F L_1(\xi)}{2(\alpha^2-\beta^2)} \;;\quad \bar{\varphi}_2(\xi) = \frac{F\gamma L_2(\xi)}{2(\alpha^2-\beta^2)} \tag{13}$$

where

$$\left.\begin{aligned}
L_1(\xi) &= \beta^2(\sigma_1\alpha^2\gamma^2+1)e^{-\xi\alpha\gamma b} - \alpha^2(\sigma_1\beta^2\gamma^2+1)e^{-\xi\beta\gamma b} \\
L_2(\xi) &= \beta(\sigma_1\alpha^2\gamma^2+1)e^{-\xi\alpha\gamma b} - \alpha(\sigma_1\beta^2\gamma^2+1)e^{-\xi\beta\gamma b}
\end{aligned}\right\} \tag{14}$$

Substitution of these values in (12), gives

$$\left.\begin{aligned}
\widehat{xx}_2 &= \frac{F\gamma^2}{2\pi(\alpha-\beta)(\alpha^2-\beta^2)}\int_0^\infty\Big[(\beta L_1+L_2)\alpha^2 e^{-\xi\alpha\gamma(b-y)} - (\alpha L_1+L_2)\beta^2 e^{-\xi\beta\gamma(b-y)}\Big]\cos \xi x \, d\xi \\
\widehat{yy}_2 &= -\frac{F}{2\pi(\alpha-\beta)(\alpha^2-\beta^2)}\int_0^\infty\Big[(\beta L_1+L_2)e^{-\xi\alpha\gamma(b-y)} - (\alpha L_1+L_2)e^{-\xi\beta\gamma(b-y)}\Big]\cos \xi x \, d\xi \\
\widehat{xy}_2 &= \frac{F\gamma}{2\pi(\alpha-\beta)(\alpha^2-\beta^2)}\int_0^\infty\Big[(\beta L_1+L_2)\alpha e^{-\xi\alpha\gamma(b-y)} - (\alpha L_1+L_2)\beta e^{-\xi\gamma\gamma(b-y)}\Big]\sin \xi x \, d\xi
\end{aligned}\right\} \tag{15}$$

The complete stresses are given by (15) together with (A). In particular, the edge stress is given by

$$(\widehat{xx})_{y=b} = (\widehat{xx}_1 + \widehat{xx}_2)_{y=b} \tag{16}$$

From (15), we have

$$(\widehat{xx}_2)_{y=b} = \frac{F\gamma^2}{2\pi(\alpha^2-\beta^2)}\int_0^\infty \{L_1 + (\alpha+\beta)L_2\} \cos \xi x \, d\xi \quad .$$

Substituting the values of L_1 and L_2 and integrating,

$$(\widehat{xx}_2)_{y=b} = \frac{F\gamma^3 b}{2\pi(\alpha^2-\beta^2)}\left[\frac{(\alpha+2\beta)(\sigma_1\alpha^2\gamma^2+1)}{x^2+\alpha^2\gamma^2 b^2} - \frac{(2\alpha+\beta)(\sigma_1\beta^2\gamma^2+1)}{x^2+\beta^2\gamma^2 b_2}\right] \tag{17}$$

From (7), (16) and (17),

$$(\widehat{xx})_{y=b} = \frac{F\gamma^2 b}{\pi(\alpha-\beta)}\left[\frac{\sigma_1\alpha^2\gamma^2+1}{x^2+\alpha^2\gamma^2 b^2} - \frac{\sigma_1\beta^2\gamma^2+1}{x^2+\beta^2\gamma^2 b^2}\right] \tag{18}$$

For an isotropic material, $\gamma \to \alpha \to \beta \to 1$ and $\sigma_1 = \sigma_2 = \sigma$,

so that
$$(\widehat{xx})_{y=b} = \frac{2Fb}{\pi} \frac{\sigma x^2 - b^2}{(x^2+b^2)^2}. \tag{18}$$

Equivalent results are given by Green (1945).

4. Isolated Transverse Force in an infinite strip. Let there be an infinite strip whose edges, given by $y = \pm b$, are parallel to one axis of symmetry, which is taken as the x-axis. Then y-axis will be parallel to the other axis of symmetry. The stress system $\widehat{xx}_1$, $\widehat{yy}_1$, $\widehat{xy}_1$ in an infinite plate due to a force F acting along the y-axis are given by the relations (A). On the edge $y = b$,

$$\left.\begin{aligned} (\widehat{xx}_1)_{y=b} &= \frac{F\gamma^3 b}{2\pi(\alpha^2+\beta^2)}\left[\frac{(\sigma_1\alpha^2\gamma^2+1)\alpha}{x^2+\alpha^2\gamma^2 b^2} - \frac{(\sigma_1\beta^2\gamma^2+1)\beta}{x^2+\beta^2\gamma^2 b^2}\right] \\ (\widehat{yy}_1)_{y=b} &= -\varphi_1(x)\,;\ (\widehat{xy}_1)_{y=b} = -\varphi_2(x), \end{aligned}\right\} \tag{20}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are defined as in (7).

On the edge $y = -b$, $\quad (yy_1)_{y=b} = \varphi_1(x), \quad (\widehat{xy}_1)_{y=b} = -\varphi_2(x),$ (21)

To nullify the above stresses on the boundaries, we are to superpose a stress system $\widehat{xx}_2$, yy_2, $\widehat{xy}$ such that

on the edge $y = b$, $\quad (yy_2)_{y=b} = \varphi_1(x), \quad (\widehat{xy}_2)_{y=b} = \varphi_2(x)$ (22)

and on the edge $y = -b$, $\quad (yy_2)_{x=-b} = -\varphi_1(x), \quad (\widehat{xy}_2)_{y=-b} = \varphi_2(x).$ (23)

Let us assume as a solution of (2),

$$G = B \sinh|\xi|\alpha\gamma y + D \sinh|\xi|\beta\gamma y \tag{24}$$

The relations (4) with the conditions either (22) or (23), give

$$\left.\begin{aligned} -\xi^2(B\sinh|\xi|\alpha\gamma b + D\sinh|\xi|\beta\gamma b) &= 2\int_0^\infty \varphi_1(x)\cos\xi x\,dx \equiv \bar{\varphi}_1(\xi) \\ \xi|\xi|\gamma(\alpha B\cosh|\xi|\alpha\gamma b + \beta D\cosh|\xi|\xi|\beta\gamma b) &= 2\int_0^x \varphi_2(x)\sinh\xi x\,dx \equiv \bar{\varphi}_2(x) \end{aligned}\right\} \tag{25}$$

These are satisfied if

$$\left.\begin{aligned} B &= \frac{\beta\cosh|\xi|\beta\gamma b\,\bar{\varphi}_1/\xi^2 + \sinh|\xi|\beta\gamma b\,\bar{\varphi}_2/(\gamma\xi|\xi|) -}{\alpha\cosh|\xi|\alpha\gamma b\sinh|\xi|\beta\gamma b - \beta\sinh|\xi|\alpha\gamma b\cos|\xi|\beta\gamma b} \\ D &= -\frac{\alpha\cosh|\xi|\alpha\gamma b\,\bar{\varphi}_1/\xi^2 + \sinh|\xi|\alpha\gamma b\,\bar{\varphi}_2/(\gamma\xi|\xi|)}{\alpha\cosh|\xi|\alpha\gamma b\sinh|\xi|\beta\gamma b - \beta\sinh|\xi|\alpha\gamma b\cosh|\xi|\beta\gamma b} \end{aligned}\right\} \tag{26}$$

Since both B and D are even functions in ξ, the relations (5) give

$$\left.\begin{aligned}
\widehat{xx}_2+\beta^2\gamma^2\widehat{yy}_2&=\frac{(\alpha^2-\beta^2)\gamma^2}{\pi}\int_0^\infty \xi^2 B(\xi)\sinh\xi\alpha\gamma y\cos\xi x\,d\xi\\
\widehat{xx}_2+\alpha^2\gamma^2\widehat{yy}^2&=-\frac{(\alpha^2-\beta^2)\gamma^2}{\pi}\int_0^\infty \xi^2 D(\xi)\sinh\xi\beta\gamma y\cos\xi x\,d\xi\\
\widehat{xy}_2&=\frac{\gamma}{\pi}\int_0^\infty \xi^2[\alpha B(\xi)\cosh\xi\alpha\gamma y+\beta D(\xi)\cosh\xi\beta\gamma y]\sin\xi x\,d\xi
\end{aligned}\right\}\tag{27}$$

To evaluated $\bar{\varphi}_1(\xi)$ and $\bar{\varphi}_2(\xi)$, we substitute the expressions for $\varphi_1(x)$ and $\varphi_2(x)$ from (20) in (25), and after integration, we have

$$\bar{\varphi}_1(\xi)=\frac{FL_1(\xi)}{2(\alpha^2-\beta^2)},\qquad \bar{\varphi}_2(\xi)=\frac{F\gamma L_2(\xi)}{2(\alpha^2-\beta^2)}\tag{28}$$

where $L_1(\xi)$ and $L_2(\xi)$ are given by (14).

Substituting these values in (27), we get finally,

$$\widehat{xx}_2+\beta^2\gamma^2\widehat{yy}_2=\frac{F\gamma^2}{2\pi}\int_0^\infty\frac{(L_1\beta\cosh\xi\beta\gamma b+L_2\sinh\xi\beta\gamma b)\sinh\xi\alpha\gamma y\cos\xi x}{\alpha\cosh\xi\alpha\gamma b\sinh\xi\beta\gamma b-\beta\sinh\xi\alpha\gamma b\cosh\xi\beta\gamma b}\,d\xi.$$

$$\widehat{xx}_2+\alpha^2\gamma^2\widehat{yy}_2=\frac{F\gamma^2}{2\pi}\int_0^\infty\frac{(L_1\alpha\cosh\xi\alpha\gamma b+L_2\sinh\xi\alpha\gamma b)\sinh\xi\beta\gamma y\cos\xi x}{\alpha\cosh\xi\alpha\gamma b\sinh\xi\beta\gamma b-\beta\sinh\xi\alpha\gamma b\cosh\xi\beta\gamma b}\,d\xi$$

$$\widehat{xy}_2=\frac{F\gamma}{2\pi(\alpha^2-\beta^2)}$$

$$\int_0^\infty\frac{\alpha\cosh\xi\alpha\gamma y\{L_1\beta\cosh\xi\beta\gamma b+L_2\sinh\xi\beta\gamma b\}-\beta\cosh\xi\beta\gamma b\{L_1\alpha\cosh\xi\alpha\gamma b+L_2\sinh\xi\alpha\gamma b}{\alpha\cosh\xi\alpha\gamma b\sinh\xi\beta\gamma b-\beta\sinh\xi\alpha\gamma b\cosh\xi\beta\gamma b}\times$$

$$\times\sin\xi x\,d\xi.\tag{29}$$

The complete stresses are given by (29) together with (A). On the edge $y=b$, $(\widehat{xx})_{y=b}=(\widehat{xx}_1)_{y=b}+(\widehat{xx}_2)_{y=b}$ and hence,

$$(\widehat{xx})_{y=b}=\frac{F\gamma^3 b}{2\pi(\alpha+\beta)}\cdot\frac{x^2\{\sigma_1\gamma^2(\alpha^2+\alpha\beta+\beta^2)+1\}+{}^2\gamma b_1(\sigma_1\gamma^2-1)}{(x^2+\alpha^2\gamma^2b^2)(x^2+\beta^2\gamma^2b^2)}$$

$$+\frac{F\gamma^2}{2\pi(\alpha^2-\beta^2)}\int_0^\infty\frac{\cos\xi x[(\sigma_1\alpha^2\gamma^2+1)\{(\beta\cosh\xi\beta\gamma b+\alpha\sinh\xi\beta\gamma b)e^{-\xi\alpha\gamma b}\sinh\xi\alpha\gamma b-\ldots}{\alpha\cosh\xi\alpha\gamma b\sinh\xi\beta\gamma b-\beta\sinh\xi\alpha\gamma b\cosh\xi\beta\gamma b}$$

$$\ldots-\beta^3\sinh\xi\beta\gamma b\}+(\sigma_1\beta^2\gamma^2+1)\{(\alpha\cosh\xi\alpha\gamma b+\beta\sinh\xi\alpha\gamma b)$$

$$e^{-\xi\beta\gamma b}\sinh\xi\beta\gamma b-\alpha^3\sinh\xi\alpha\gamma b\}]d\xi.\tag{29·1}$$

5. Isolated Couple acting at an internal point in the infinite strip. If the plate be unbounded in all directions, the stresses $\widehat{xx}_1$, $\widehat{yy}_1$ and $\widehat{xy}_1$ due to a couple of moment M are given by the relations (B). On the edge $y=b$,

$$\left.\begin{aligned}
(\widehat{xx}_1)_{y=b} &= -\frac{M\gamma^2 bx}{\pi(\alpha^2-\beta^2)}\left[\frac{\alpha(\sigma_1\alpha^2\gamma^2+1)}{(x^2+\alpha^2\gamma^2b^2)^2}-\frac{\beta(\sigma_1\beta^2\gamma^2+1)}{(x^2+\beta^2\gamma^2b^4)^2}\right]\\
(\widehat{yy}_1)_{y=b} &= \frac{M\gamma bx}{\pi(\alpha^2-\beta^2)}\left[\frac{(\sigma_1\alpha^2\gamma^2+1)\beta}{(x^2+\alpha^2\gamma^2b^2)^2}-\frac{(\sigma_1\beta^2\gamma^2+1)\alpha}{(x^2+\beta^2\gamma^2b^2)^2}\right]\equiv\psi_1(x)\\
(\widehat{xy}_1)_{y=b} &= \frac{M\gamma}{2\pi(\alpha^2-\beta^2)}\left[\frac{\beta(\sigma_1\alpha^2\gamma^2+1)(x^2-\alpha^2\gamma^2b^2)}{(x^1+\alpha^2\gamma^2b^2)^2}-\frac{\alpha(\sigma_1\beta^2\gamma^2+1)(x^2-\beta^2\gamma^2b^2)}{(x^2+\beta^2\gamma^2b^2)^2}\right]\equiv\psi_2(x)
\end{aligned}\right\}\quad(30)$$

and on the edge $y=-b$,

$$(\widehat{yy}_1)_{y=-b}=-\psi_1(x),\quad(\widehat{xy}_1)_{y=-b}=\psi_2(x).\tag{31}$$

We superpose a stress system $\widehat{xx}_2$, $\widehat{yy}_2$, $\widehat{xy}_2$ such that

on the edge $y=b$, $\quad(\widehat{yy}_2)_{y=b}=-\psi_1(x),\quad(\widehat{xy}_2)_{y=b}=-\psi_2(x),$ (32)

and on the edge $y=-b$, $\quad(\widehat{yy}_2)_{y=-b}=\psi_1(x),\quad(\widehat{xy}_2)_{y=-b}=-\psi_2(x)$ (33)

Let us assume $\quad G=B\sinh|\xi|\alpha\gamma y+D\sinh|\xi|\beta\gamma y$ (34)

The second and third relations of (4) with the boundary conditions either (32) or (33), give,

$$\left.\begin{aligned}
\xi^2(B\sinh|\xi|\alpha\gamma b+D\sinh|\xi|\beta\gamma b) &= 2i\int_0^\infty\psi_1(x)\sin\xi x\,dx\equiv\bar\psi_1(\xi)\\
\xi|\xi|i\gamma(\alpha B\cosh|\xi|\alpha\gamma b+\beta D\cosh|\xi|\beta\gamma b) &= -2\int_0^\infty\psi_1(x)\cos\xi x\,dx\equiv-\bar\psi_2(\xi).
\end{aligned}\right\}\quad(35)$$

These are satisfied if

$$\left.\begin{aligned}
B &= \frac{\beta\cosh|\xi|\beta\gamma b\,\bar\psi_1/\xi^2+\sinh|\xi|\beta\gamma b\,\psi_2/(i\gamma|\xi|\xi)}{\beta\cosh|\xi|\beta\gamma b\sinh|\xi|\alpha\gamma b-\alpha\sinh|\xi|\beta\gamma b\cosh|\xi|\alpha\gamma b}\\
D &= -\frac{\alpha\cosh|\xi|\alpha\gamma b\,\bar\psi_1/\xi^2+\sinh|\xi|\alpha\gamma b\,\bar\psi_2/(i\gamma|\xi|\xi)}{\beta\cosh|\xi|\beta\gamma b\sinh|\xi|\alpha\gamma b-\alpha\sinh|\xi|\beta\gamma b\cosh|\xi|\alpha\gamma b}.
\end{aligned}\right\}\quad(36)$$

Since both B and D are odd functions in ξ, (6)' gives,

$$\left.\begin{aligned}
\widehat{xx}_2+\beta^2\gamma^2\widehat{yy}_2 &= -\frac{i(\alpha^2-\beta^2)\gamma^2}{\pi}\int_0^\infty\xi^2B(\xi)\sinh\xi\alpha\gamma y\sin\xi x\,d\xi.\\
\widehat{xx}_2+\alpha^2\gamma^2\widehat{yy}_2 &= \frac{i(\alpha^2-\beta^2)\gamma^2}{\pi}\int_0^\infty\xi^2D(\xi)\sinh\xi\beta\gamma y\sin\xi x\,d\xi.\\
\widehat{xy}_2 &= \frac{i\gamma}{\pi}\int_0^\infty\xi^2[\alpha B(\xi)\cosh\xi\alpha\gamma y+\beta D(\xi)\cosh\xi\beta\gamma y]\cos\xi x\,d\xi
\end{aligned}\right\}\quad(37)$$

To evaluate $\bar\psi_1(\xi)$ and $\bar\psi_2(\xi)$, we substitute the expressions for $\psi_1(x)$ and $\psi_2(x)$ in (35) from (30), and after integration, we have,

$$\bar{\psi}_1(\xi) = \frac{iM\xi}{2(\sigma^2-\beta^2)} L_1(\xi), \qquad \bar{\psi}_2(\xi) = -\frac{M\gamma\xi}{2(\sigma^2-\beta^2)} L_2(\xi), \tag{38}$$

where $L_1(\xi)$ and $L_2(\xi)$ are given by (14). Substituting these values in (37), we get

$$\left.\begin{aligned}
\widehat{xx}_2+\beta^2\gamma^2\widehat{yy}_2 &= \frac{M\gamma^2}{2\pi}\int_0^\infty \frac{(L_1\beta\cosh\xi\beta\gamma b+L_2\sinh\xi\beta\gamma b)\xi\sinh\xi\alpha\gamma y\sin\xi x}{\beta\cosh\xi\beta\gamma b\sinh\xi\alpha\gamma b-\sigma\sinh\xi\beta\gamma b\cosh\xi\alpha\gamma b}\,d\xi. \\
\widehat{xx}_2+\alpha^2\gamma^2\widehat{yy}_2 &= \frac{M\gamma^2}{2\pi}\int_0^\infty \frac{(L_1\alpha\cosh\xi\alpha\gamma b+L_2\sinh\xi\alpha\gamma b)\xi\sinh\xi\beta\gamma y\sin\xi x}{\beta\cosh\xi\beta\gamma b\sinh\xi\alpha\gamma b-\alpha\sinh\xi\beta\gamma b\cosh\xi\alpha\gamma b}\,d\xi \\
\widehat{xy}_2 &= -\frac{M\gamma}{2\pi(\sigma^2-\beta^2)}\int_0^\infty \frac{[\alpha\xi\cosh\xi\alpha\gamma\, y(L_1\beta\cosh\xi\beta b+D_2\sinh\xi\beta\gamma b)-\ldots}{\beta\cosh\xi\beta\gamma b\sinh\xi\alpha\gamma b-\alpha\sinh\xi\beta\gamma b\cosh\xi\alpha\gamma b}\,. \\
&\qquad \ldots-\beta\xi\cosh\xi\beta\gamma y(D_1\alpha\cosh\xi\alpha b+L_2\sinh\xi\alpha\gamma b)]\cos\xi x\,d\xi.
\end{aligned}\right\} \tag{39}$$

The complete stresses are given by (39)and (B).

When $y = b$ and $x\to\infty$, from (39), (Carslaw, 1921)

$$\widehat{xx}_2+\beta^2\gamma^2\widehat{yy}_2\to\frac{3M}{4b^2} \text{ and } \widehat{xx}_2+\alpha^2\gamma^2\widehat{yy}_2\to\frac{3M}{4b^2}$$

so that

$$\widehat{xx}_2\to\frac{3M}{4b^2} \text{ and } \widehat{yy}_2\to 0.$$

Also since the complete stresses $\widehat{yy}$ and $\widehat{xy}$ on the edge $y = b$ vanish and $\widehat{xx}_1\to 0$, we have the edge stress at a great distance

$$\widehat{xx} = \frac{3M}{4b^2}\,. \tag{40}$$

Since this is independent of α, β and γ, we see that the aeolotropic plate behaves as an isotropic one at a great distance. The edge stress (40) for an isotropic material was given by Sen (1942).

At any point on the edge $y = b$, the stress is given by

$$(\widehat{xx})_{y=b}=(\widehat{xx}_1)_{y=b}+(\widehat{xx}_2)_{y=b} \tag{41}$$

Substituting $$x = x_1\beta\gamma b \text{ and } \xi\beta\gamma b = \xi_1 \tag{42}$$

we have

$$\frac{\pi b^2}{M}(\widehat{xx})_{y=b} = -\frac{x_1}{\alpha^2-\beta^2}\left[\frac{\sigma_1\alpha^2\gamma^2+1}{\beta^4(x_1{}^2+\sigma^4)^2}-\frac{\sigma_1\gamma^2+\alpha^2}{(x_1{}^2+1)^2}\right]+\frac{\sigma_1\beta^2\gamma^2+1}{2(1-\beta^4)}\int_0^\infty f(\xi_1)\sin\xi_1 x_1\,d\xi_1 \tag{43}$$

where,

$$\begin{aligned}
f(\xi_1) = \xi_1[\{\alpha^3-d\alpha\, e^{-\alpha^2\xi_1}(\beta^2\cosh\xi_1+\sinh\xi_1)\}\sinh\alpha^2\xi_1+\{\beta^3 d-\beta\, e^{-\xi_1} \\
\times(\alpha^2\cosh\alpha^2\xi_1+\sinh\sigma^2\xi_1)\}\sinh\xi_1]/(\alpha\cosh\alpha^2\xi_1\sinh\xi_1-\beta\sinh\alpha^2\xi_1\cosh\xi_1)
\end{aligned} \tag{44}$$

and

$$d = \frac{\sigma_1\alpha^2\gamma^2+1}{\sigma_1\beta^2\gamma^2+1} \tag{45}$$

The integral in the right side of (48) can be evaluated for given values of x_1 by the method given by Filon (1929). This method requires that $f(\xi_1)$ must have the limit as ξ_1 tends to zero. This condition is not satisfied by $f(\xi_1)$, as

$$f(\xi_1) \sim \frac{3(1-\beta^4 d)}{\xi_1} + O(\xi_1) \tag{46}$$

when ξ_1 is small. If we define

$$F(\xi_1) = f(\xi_1) - 3(1-\beta^4 d)\,\frac{1+\xi_1}{\xi_1}\,e^{-\xi_1}, \tag{47}$$

then $F(\xi_1)$ satisfied the required condition. Hence we can write

$$I(x_1) \equiv \int_8^\infty f(\xi_1)\sin\xi_1 x_1\,d\xi_1 = 3(1-\beta^4 d)\left\{\frac{x_1}{1+x_1^2} + \tan^{-1}x_1\right\} + \int_0^\infty F(\xi_1)\sin\xi_1 x_1\,d\xi_1 \tag{48}$$

For a specimen of spruce wood, when the grain of the wood is perpendicular to the edges of the strip, we take the elastic constants as follows; (Green, 1939)

$$\left.\begin{aligned} &\frac{1}{E_1} = 15{\cdot}5, \quad \frac{1}{E_2} = 0{\cdot}587, \quad \frac{\sigma}{E_2} = 0{\cdot}33, \quad \frac{1}{\mu_1} = 11{\cdot}5 \\ \text{so that} \quad &\alpha = 1{\cdot}8186, \;\; \beta = 0{\cdot}5525, \;\; \gamma = 0{\cdot}4411, \;\; \sigma_1 = 0{\cdot}562 \end{aligned}\right\} \tag{49}$$

With these values of the constants, the effective upper limit in the integral in (48) may be taken as 10. We are also to note, while evaluating the integral numerically, that, for $\xi_1 \geq 4$

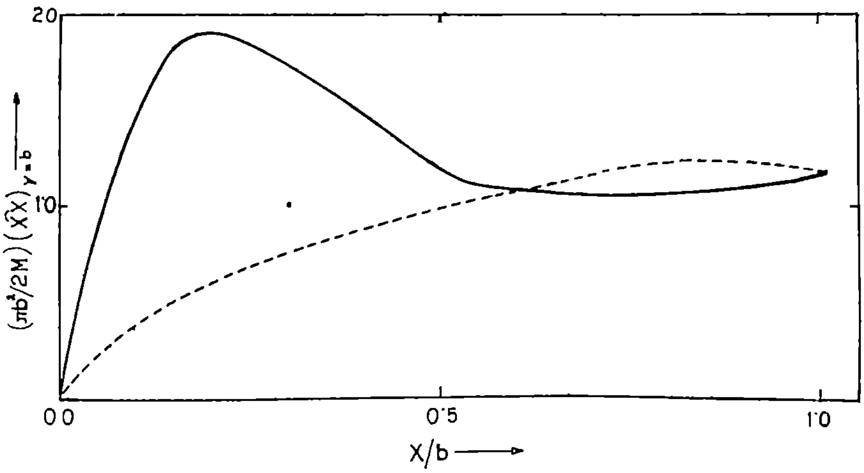
$$f(\xi_1) \sim \frac{[\alpha^3 - \frac{1}{2}\alpha + \beta]\xi_1}{\alpha\sinh\xi_1 - \beta\cosh\xi_1} \tag{50}$$

The results of calculation for the edge stress is given in Table I In the figure I, the continuous curve gives the values of $\frac{\pi b^2}{2M}\widehat{(xx)}_{y=b}$ for different values of x/b. The dotted curve gives the corresponding values when the material is isotropic and was given by Sen (1942).

Table I

x_1	$\pi/36$	$\pi/18$	$\pi/9$	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	π	$4\pi/3$
x/b	0.021	0.042	0.058	0.128	0.191	0.255	0.383	0.510	0.766	1.021
$\frac{\pi b^2}{2M}\widehat{(xx)}_{y=b}$	0.4802	0.9256	1.5964	1.9150	1.9703	1.7881	1.5497	1.1678	1.1607	1.1678

Figure I



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A NOTE ON CLASSIFICATION OF QUADRICS BY THE MATRIX METHOD

By

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1. Quadrics are generally classified by considering the roots of the discriminating cubic or the nature of the centre and central planes. The object of this note is to obtain systematic criteria for various classes of quadrics with the help of elementary results in the Matrix Theory.

2. The following elementary notions and results of Matrix Theory will be required for the investigation in question:

I. If $A = [a_{ij}]$, $B = [b_{kl}]$ be two $m \times n$ and $n \times p$ matrices, then their product $AB = C$ is a $m \times p$ matrix where

$$C = [c_{il}], \quad c_{il} = \sum_{j=1}^{n} a_{ij} b_{jl}.$$

II. *Rank.* If *every* minor of order $(\varrho+1)$ of a matrix is zero and *some* minor of order ϱ is $\neq 0$, then ϱ is said to be the rank of matrix. The symbol $\varrho(A)$ will denote the rank of the matrix A.

III. If A, B are any two matrices such that there exists a non-singular matrix P with the property $P'AP = B$, where P' is the transpose of P, then A, B are said to be *congruent*.

The ranks of two congruent matrices are the same.

A matrix P is non-singular if and only if its determinant $|P| \neq 0$.

IV. The determinant of the product of two matrices is the product of their determinants, *i.e.*, $|AB| = |A| \cdot |B|$.

3. **The matrix associated with a quadric.** The matrix

$$Q \equiv \begin{bmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{bmatrix}$$

will be said to be associated with the polynomial $F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d$. Also by the rank and determinant of $F(x, y, z)$, we shall mean the rank and determinant of the matrix Q.

We shall first prove the following Lemma which is fundamental:

Under every change of rectangular cartesian axes, both the rank and the determinant of $F(x, y, z)$ are invariant.

Let the equations of transformation of coordinates from x, y, z to X, Y, Z be

$$\begin{aligned} x &= l_1X + m_1Y + n_1Z + p_1, \\ y &= l_2X + m_2Y + n_2Z + p_2, \\ z &= l_3X + m_3Y + n_3Z + p_3, \end{aligned}$$

so that the matrix

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

is orthogonal.

The polynomial $F(x, y, z)$ and the equations of transformation can be written in matrix forms as

$$F(x, y, z) = [x \ y \ z \ 1] \begin{bmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

and

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

The matrix associated with the transform of $F(x, y, z)$, *viz.*,

$$\begin{bmatrix} l_1 & l_2 & l_3 & 0 \\ m_1 & m_2 & m_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ p_1 & p_2 & p_3 & 1 \end{bmatrix} \begin{bmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)$$

is congruent to the matrix Q, associated with $F(x, y, z)$.

Also the determinants of the premultiplier and the postmultiplier of Q in (1) are each equal to unity.

Hence the lemma follows.

4. We know that there exists an orthogonal matrix

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

such that the corresponding linear transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

transforms $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ to $\lambda_1x^2+\lambda_2y^2+\lambda_3z^2$ where λ_1, λ_2, λ_3 are the characteristic roots of the matrix

$$P \equiv \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}.$$

This rotation of axes will throughout be denoted by the symbol R.

5. We note that $\varrho(Q) \leq 4$, $\varrho(P) \leq 3$ and $\varrho(P) \leq \varrho(Q)$.

The following cases arise:

I. $\varrho(Q) = 4$, $\varrho(P) = 3$.

By R, $F(x, y, z)$ will be transformed to a polynomial of the form.

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+2u_1x+2v_1y+2w_1z+d_1$$

where none of λ_1, λ_2, λ_3 is zero.

Shifting the origin to $\left(-\frac{u_1}{\lambda_1}, -\frac{v_1}{\lambda_2}, -\frac{w_1}{\lambda_3}\right)$

it takes the form, $\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+k$.

As its rank is 4, $k \neq 0$.

Thus the quadric is central—an ellipsoid, a hypeboloid of one sheet, a hyperboloid of two sheets, or a sphere.

Also as the determinant is an invariant,

$$|Q| = \lambda_1\lambda_2\lambda_3k = |P|k \text{ or } k = |Q|/|P|.$$

II. $\varrho(Q) = 4$, $\varrho(P) = 2$.

By R, the transform is of the form

$$\lambda_1x^2+\lambda_2y^2+2u_1x+2v_1y+2w_1z+d_1.$$

Shifting the origin to $\left(-\frac{u_1}{\lambda_1}, -\frac{v_1}{\lambda_2}, 0\right)$

we obtain $\lambda_1x^2+\lambda_2y^2+2w_1z+k$.

As its rank is 4, $w_1 \neq 0$.

Shifting the origin to $(0, 0, -w_1/k)$

we obtain $\lambda_1x^2+\lambda_2y^2+2w_1z$

so that the surface is a *paraboloid*,

Also $$|Q| = -\lambda_1\lambda_2 w_1^2 \text{ or } w_1 = (-|Q|/(\lambda_1\lambda_2))^{\frac{1}{2}}.$$

III. $$\varrho(Q) = 4,\ \varrho(P) = 1.$$

By R, the transform takes the form,

$$\lambda_1 x^2 + 2u_1 x + 2v_1 y + 2w_1 z + d_1.$$

Shifting the origin to $(-u_1/\lambda_1,\ 0,\ 0)$, we obtain

$$\lambda_1 x^2 + 2v_1 y + 2w_1 z + d_1$$

whose determinant is 0. As $\varrho(Q) = 4$, we see that this case is not possible.

IV. $$\varrho(Q) = 3,\ \varrho(P) = 0.$$

By R, we obtain $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + 2u_1 x + 2v_1 y + 2w_1 z + d_1$.

As in case I, this can be further transformed to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + k,$$

whose rank being 0, we have $k = 0$.

Thus the surface is a *cone*

V. $$\varrho(Q) = 3,\ \varrho(P) = 2.$$

By R, the transform is

$$\lambda_1 x^2 + \lambda_2 y^2 + 2u_1 x + 2v_1 y + 2w_1 z + d_1.$$

Shifting the origin to $(-u_1/\lambda_1,\ -v_1/\lambda_2,\ 0)$ we obtain

$$\lambda_1 x^2 + \lambda_2 y^2 + 2w_1 z + k.$$

Its determinant $= -\lambda_1\lambda_2 w_1^2 = 0$. Thus $w_1 = 0$.

Also, since $\varrho(Q) = 3$, $\therefore\ k \neq 0$.

Therefore the surface is a cylinder, elliptic or hyperbolic according as λ_1 and λ_2 have the same or opposite signs.

VI. $$\varrho(Q) = 3,\ \varrho(P) = 1.$$

By R, the transform is

$$\lambda_1 x^2 + 2u_1 x + 2v_1 y + 2w_1 z + d_1.$$

Shifting the origin to $(-v_1/\lambda_1,\ 0,\ 0)$ we obtain

$$\lambda_1 x^2 + 2v_1 y + 2w_1 z + k.$$

Here v_1 and w_1 cannot both be zero, for $\varrho(Q) = 3$.

If $v_1 \neq 0$, shifting the origin to $(0,\ -k/2v_1,\ 0)$ we obtain

$$\lambda_1 x^2 + 2v_1 y + 2w_1 z.$$

Finally employing the transformation

$$x = X,\ y = Y\cos\theta - Z\sin\theta,\ z = Y\sin\theta + Z\cos\theta,$$

where $\tan\theta = v_1/w_1$, we obtain $\lambda_1 X^2 + 2v_2 Y$

so that the surface is a *parabolic cylinder*,

VII. $\varrho(Q) = 2,\ \varrho(P) = 2.$

By R, we obtain $\lambda_1 x^2 + \lambda_2 y^2 + 2v_1 x + 2v_1 y + 2w_1 z + d_1$
and again by shifting the origin to $(-u_1/\lambda_1,\ -v_1/\lambda_2,\ 0)$
we obtain $\lambda_1 x^2 + \lambda_2 y^2 + 2w_1 z + k$

As its rank is 2, we must have $w_1 = 0,\ k = 0.$

Thus the surface is a pair of *intersecting planes.*

VIII. $\varrho(Q) = 2,\ \varrho(P) = 1.$

By R, and then by shifting the origin to $(-u/\lambda_1,\ 0,\ 0)$, the transform is

$$\lambda_1 x^2 + 2v_1 y + 2w_1 z + k$$

As $\varrho(Q) = 2$, every third order minor of the matrix associated with the transform is zero and as such, we have $v_1 = 0,\ w_1 = 0.$

Also $k \neq 0.$

Thus the surface is a pair of *parallel planes.*

IX. $\varrho(Q) = 1,\ \varrho(P) = 1.$

In this case $F(x, y, z)$ transforms to $\lambda_1 x^2$ so that the surface is a pair of *coincident planes.*

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DERIVATION OF LAGRANGIAN AND HAMILTONIAN EQUATIONS OF MOTION FROM THE DYNAMICS IN HYPERSPACE*

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Introduction. The conception of multidimensional space associated with Dynamics is endowed with greater reality than that in the Pure Geometry itself. When in a dynamical system the quantities involved allow themselves to be fully determined by more than three measurable data, it is often convenient to employ the notations and processes of the Geometry of Hyperspace. Thus a system of $n/3$ individual particles in motion in ordinary space, represented by n coordinates, is assimilated in a single 'generalized' particle moving in a euclidean n-space. When the configuration of the system can be determined by knowing a smaller number r of a set of independent quantities, called generalized coordinates, certain geometrical conditions expressed by means of $(n-r)$ equations are then involved and the generalized particle may be regarded as moving on an r-dimensional variety in n-space. The object of this paper is to study this kind of motion in hyperspace by a new vectorial method, based on the properties of determinants and to derive therefrom the Lagrangian and Hamiltonian equations of motion.

2. *r*-dimensional variety. Let the r-dimensional variety be referred to an orthogonal set of n unit vectors [1] I_a passing through an origin 0 in a euclidean n-space. The position of a point on the variety is then given by the radius vector[1]

$$V=\sum_{a=1}^{n} v^a I_a, \tag{2.1}$$

where the v's are scalar functions of r parameters $(u^1, u^2, \ldots u^r)$. Fixing upon a point P on the variety corresponding to definite values of the parameters u^k, the tangent subspace at the point P is spanned by the r vectors

$$V_k=\sum_{a=1}^{n}\frac{\partial v^a}{\partial u^k} I_a \qquad (k=1, 2, \ldots\ldots r) \tag{2.2}$$

The osculating subspace of the second order at the point P is spanned by the $\binom{r+2}{2}-1$ vectors V_k and $V_{k_1k_2}$, where

$$V_{k_1k_2}=\sum_{a=1}^{n}\frac{\partial^2 v^a}{\partial u^{k_1}\partial u^{k_2}} I_a; \tag{2.3}$$

and so on for higher order osculating subspaces.

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* This paper was read at the Calcutta Session of the Indian Science Congress, 1952.

[1] Capital letters of English alphabet, the letters O, P, Q, T being excepted, will denote vectors.

3. Velocity and acceleration vectors of a particle. At time t let the particle be at a point Q on the variety given by the vector

$$X = V(q^1, q^2, \ldots q^r) \tag{3.1}$$

where the position coordinates q^i are functions of t. Differentiating the above with respect to t, the velocity vector is expressible in the form

$$\dot{X} = \sum_{h=1}^{r} X_h \dot{q}^h, \tag{3.2}$$

where the system of r vectors X_h forms the tangent subspace of the variety at Q. Adopting, henceforth, the usual convention for repeated indices[2] as in tensor analysis, the acceleration at time t is expressed by the vector

$$\dot{X} = X_h \dot{q}^h + X_{h_1h_2}\dot{q}^{h_1}\dot{q}^{h_2}, \tag{3.3}$$

where the system of $\binom{r+2}{2} - 1$ vectors X_h and $X_{h_1h_2}$ forms the osculating subspace of the second order of the variety at Q.

4. Momentum vector and the kinetic energy. Referred to the fundamental basis I_α let us write $\dot{X}$ in the form

$$X = I_\alpha \left\{ \begin{matrix} I_\alpha \\ \dot{X} \end{matrix} \right\}. \tag{4.1}$$

where the symbol $\{\,\}$ denotes scalar product of the vectors involved, We next define a vector $\dot{Y}$ whose components referred to the fundamental basis are

$$m_1\left\{ \begin{matrix} 1_1 \\ \dot{X} \end{matrix} \right\},\ m_2\left\{ \begin{matrix} I_2 \\ \dot{X} \end{matrix} \right\}, \ldots\ldots m_n\left\{ \begin{matrix} I_n \\ \dot{X} \end{matrix} \right\}, \tag{4.2}$$

m's being arbitrary quantities playing the role of 'mass' of the generalized particle We shall take the vector $\dot{Y}$ to represent the momentum[3] of the particle at time t. Denoting the vectors m_1I_1, m_2I_2, $\ldots\ldots m_nI_n$ by M_1, M_2, $\ldots\ldots M_n$ respectively we may write

$$\dot{Y} = I_\alpha \left\{ \begin{matrix} M_\alpha \\ \dot{X} \end{matrix} \right\} = M_\alpha \left\{ \begin{matrix} 1_\alpha \\ \dot{X} \end{matrix} \right\}. \tag{4.3}$$

Any two vectors related as above will be called 'adjoint' vectors.

Thus

$$Y_k = I_\alpha \left\{ \begin{matrix} M_\alpha \\ X_k \end{matrix} \right\} = M_\alpha \left\{ \begin{matrix} 1_\alpha \\ X_k \end{matrix} \right\}, \tag{4.4}$$

$$Y_{h_1h_2} = I_\alpha \left\{ \begin{matrix} M_\alpha \\ X_{h_1h_2} \end{matrix} \right\} = M_\alpha \left\{ \begin{matrix} 1_\alpha \\ X_{h_1h_2} \end{matrix} \right\} \tag{4.5}$$

denote vectors adjoint to X_k and $X_{h_1h_2}$ respectively.

Referring to (3.2) we have for the momentum vector the expression

$$\dot{Y} = Y_k \dot{q}^k. \tag{4.6}$$

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[2] α runs from 1 to n; h k, l, from 1 to r, and ω ranges over the values 0, 1, 2,......r.

[3] If the m's be equal by threes the case of a system of particles moving in ordinary space presents itself.

Hence the kinetic energy T of the particle at time t is given by

$$T = \tfrac{1}{2}\left\{ \begin{matrix} \dot{X} \\ \dot{Y} \end{matrix} \right\} = \tfrac{1}{2}\left\{ \begin{matrix} X_k \\ Y_l \end{matrix} \right\} \dot{q}^k \dot{q}^l, \tag{4.7}$$

where

$$\left\{ \begin{matrix} X_k \\ Y_l \end{matrix} \right\} = \left\{ \begin{matrix} X_l \\ Y_k \end{matrix} \right\}$$

5. Reciprocal vectors in the tangent subspace and an alternative expression for the kinetic energy. Let us now consider the symmetrical determinant of order r, whose $(k, l)^{th}$ element is the scalar product $\left\{ \begin{matrix} X_k \\ Y_l \end{matrix} \right\}$ of the vectors X_k and Y_l. Representing this determinant in the compact notation

$$\Delta_r = \left\{ \begin{matrix} X_1 \; X_2 \ldots\ldots X_r \\ Y_1, \; Y_2 \ldots\ldots Y_r \end{matrix} \right\} \tag{5.1}$$

we call it the scalar determinant of two sets of vectors X_k and Y_l.

Remembering the relations

$$\left\{ \begin{matrix} X_k \\ Y_l \end{matrix} \right\} \left\{ \begin{matrix} X_k \\ Y_h \end{matrix} \right\}_c = \left\{ \begin{matrix} \Delta_r \text{ if } h = l \\ 0 \text{ if } h \neq l \end{matrix} \right\}, \tag{5.2}$$

where $\left\{ \begin{matrix} X_k \\ Y_h \end{matrix} \right\}_c$ denotes the minor of $\left\{ \begin{matrix} X_k \\ Y_h \end{matrix} \right\}$ in Δ_r preceded by the proper sign, we define the system of r vectors $X_{(h)}$ and its adjoint $Y_{(h)}$ by means of the equations

$$\left. \begin{aligned} X_{(h)} &= X_k \left\{ \begin{matrix} X_k \\ Y_h \end{matrix} \right\}_c \Big/ \Delta_r, \\ Y_{(h)} &= Y_k \left\{ \begin{matrix} X_h \\ Y_k \end{matrix} \right\}_c \Big/ \Delta_r, \end{aligned} \right\} \tag{5.3}$$

so that

$$\left\{ \begin{matrix} X_{(h)} \\ Y_l \end{matrix} \right\} = \left\{ \begin{matrix} X_l \\ Y_{(h)} \end{matrix} \right\} = \left\{ \begin{matrix} 1 \text{ if } h = l \\ 0 \text{ if } h \neq l \end{matrix} \right\} \tag{5.4}$$

We shall call the pair X_l, $Y_{(h)}$ or the pair Y_l, $X_{(h)}$ mutually 'reciprocal' sets of vectors.

From (4.7) we have

$$\frac{\partial T}{\partial \dot{q}^l} = \left\{ \begin{matrix} X_k \\ Y_l \end{matrix} \right\} \dot{q}^k = \left\{ \begin{matrix} \dot{X} \\ Y_l \end{matrix} \right\} = \left\{ \begin{matrix} \dot{Y} \\ X_l \end{matrix} \right\} \tag{5.5}$$

Denoting the above by p_l, the generalized components of momentum, we can re-write (3.2) in the form

$$\left. \begin{aligned} \dot{X} &= X_{(l)} p_l, \\ \text{also} \qquad \dot{Y} &= Y_{(l)} p_l, \end{aligned} \right\} \tag{5.6}$$

The kinetic energy[4] T has therefore the alternative form

$$T' = \tfrac{1}{2}\left\{\begin{matrix} X_{(h)} \\ Y_{(l)} \end{matrix}\right\} p_h p_l, \tag{5.7}$$

where

$$\left\{\begin{matrix} X_{(h)} \\ Y_{(l)} \end{matrix}\right\} = \left\{\begin{matrix} X_{(l)} \\ Y_{(h)} \end{matrix}\right\} = \left\{\begin{matrix} X_l \\ Y_h \end{matrix}\right\}_c \Big/ \Delta_r.$$

Differentiating (4.7) and (5.7) with respect to q^k, we get

$$\frac{\partial T}{\partial q^k} = \left\{\begin{matrix} X_{hk} \\ Y_l \end{matrix}\right\} \dot{q}^h \dot{q}^l, \tag{5.8}$$

$$\frac{\partial T'}{\partial q^k} = \left\{\begin{matrix} X_{(h)} \\ Y_{(l)k} \end{matrix}\right\} p_h p_l. \tag{5.9}$$

Again, differentiating T' with respect to p_k, we have

$$\frac{\partial T'}{\partial p_k} = \left\{\begin{matrix} X_{(h)} \\ Y_{(k)} \end{matrix}\right\} p_h = \left\{\begin{matrix} \dot{X} \\ Y_{(k)} \end{matrix}\right\}, \tag{5.10}$$

which, in view of (3.2), is equivalent to

$$\frac{\partial T'}{\partial p_k} = \dot{q}^k. \tag{5.11}$$

6. Vectors with double subscripts orthogonal to the tangent subspace.

We consider now the nonvanishing scalar determinant[5]

$$\Delta_{rr} = \left\{\begin{matrix} X_1 X_2 \ldots X_r X_{11} X_{12} \ldots X_{h_1h_2} \ldots X_{rr} \\ Y_1 Y_2 \ldots Y_r Y_{11} Y_{12} \ldots Y_{h_1h_2} \ldots Y_{rr} \end{matrix}\right\} \tag{6.1}$$

of order $\binom{r+2}{2} - 1$, where the vectors $X_{h_1h_2}$ and $Y_{h_1h_2}$ with double subscripts are written with $h_1 \leq h_2$. Let us introduce the system of vectors defined by

$$\overset{*}{X}_{h_1h_2} = X_{h_1h_2} - X_h \left\{\begin{matrix} X_{h_1h_2} \\ Y_{(h)} \end{matrix}\right\} \tag{6.2}$$

and its adjoint

$$\overset{*}{Y}_{h_1h_2} = Y_{h_1h_2} - Y_h \left\{\begin{matrix} Y_{h_1h_2} \\ X_{(h)} \end{matrix}\right\}. \tag{6.3}$$

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[4] The expression for the kinetic energy is of the same form that represents the arc of a curve in geometry. Associating this expression with the line-element of some curved manifold and applying tensor methods J. L. Synge (1927) has developed his 'Geometry of Dynamics'. With the same idea but with older methods Hertz (1894) studied the principles of mechanics. In these methods the fundamental variety on which the motion of the generalized particle is represented occupies a place of secondary importance.

[5] We tacitly assume here that n is greater than its order and the vectors involved are all mutually independent; otherwise, the nonvanishing determinant of maximum order is to be taken.

Since

$$\left\{\begin{matrix} \overset{*}{X}_{h_1h_2} \\ Y_{(k)} \end{matrix}\right\} = \left\{\begin{matrix} \overset{*}{Y}_{h_1h_2} \\ X_{(k)} \end{matrix}\right\} = 0$$

it follows that $\overset{*}{X}_{h_1h_2}$ is orthogonal to each member of Y_k and $\overset{*}{Y}_{h_1h_2}$ is orthogonal to each member of X_k.

Replacing each of the vectors $X_{h_1h_2}$ in the first row of Δ_{rr} by $\overset{*}{X}_{h_1h_2}$ and each of the vectors $Y_{h_1h_2}$ in the second row by $\overset{*}{Y}_{h_1h_2}$, we notice that the value of the determinant remains unchanged while it resolves into the product form

$$\Delta_{rr} = \Delta_r \, \overset{*}{\Delta}_r, \qquad (r \text{ not summed}).$$

where

$$\overset{*}{\Delta}_r = \left\{\begin{matrix} \overset{*}{X}_{11} & \overset{*}{X}_{12} \ldots \overset{*}{X}_{h_1h_2} \ldots \overset{*}{X}_{rr} \\ \overset{*}{Y}_{11} & \overset{*}{Y}_{12} \ldots \overset{*}{Y}_{h_1h_2} \ldots \overset{*}{Y}_{rr} \end{matrix}\right\} \tag{6.4}$$

By means of this determinant of order $\binom{r+1}{2}$ we may define the systems of vectors[6] $\overset{*}{X}_{(k_1k_2)}$ and $\overset{*}{Y}_{(k_1k_2)}$, such that

$$\left\{\begin{matrix} \overset{*}{X}_{(k_1k_2)} \\ \overset{*}{Y}_{h_1h_2} \end{matrix}\right\} = \left\{\begin{matrix} \overset{*}{X}_{h_1h_2} \\ \overset{*}{Y}_{(k_1k_2)} \end{matrix}\right\} = \left\{\begin{matrix} 1 \text{ if } h_1, h_2 = k_1, k_2 \\ 0 \text{ if } h_1, h_2 \neq k_1, k_2 \end{matrix}\right\} \tag{6.5}$$

The explicit forms of these vectors, reciprocal to $\overset{*}{Y}_{h_1h_2}$ aad $\overset{*}{X}_{h_1h_1}$ respectively, are given by

$$\begin{aligned} \overset{*}{X}_{(k_1k_2)} &= \overset{*}{X}_{h_1h_2} \left\{\begin{matrix} \overset{*}{X}_{h_1h_2} \\ \overset{*}{Y}_{k_1k_2} \end{matrix}\right\}_c \Big/ \overset{*}{\Delta}_r, \\ & \qquad\qquad\qquad\qquad h_1 \leq h_2 \\ \overset{*}{Y}_{(k_1k_2)} &= \overset{*}{Y}_{h_1h_2} \left\{\begin{matrix} \overset{*}{X}_{k_1k_2} \\ \overset{*}{Y}_{h_1h_2} \end{matrix}\right\}_c \Big/ \overset{*}{\Delta}_r, \end{aligned} \tag{6.6}$$

where the suffix c has the same significance as in (5.2) and refers to the determinant $\overset{*}{\Delta}_r$ in (6.4).

Let L be any vector in the osculating subspace of the second order of the variety at the point Q, then by virtue of (5.4) and (6.5), it is expressible in the form

$$L = X_h \left\{\begin{matrix} Y_{(h)} \\ L \end{matrix}\right\} + \overset{*}{X}_{h_1h_2} \left\{\begin{matrix} \overset{*}{Y}_{(h_1h_2)} \\ L \end{matrix}\right\} \tag{6.7}$$

or, in the equivalent form

$$L = X_{(h)} \left\{\begin{matrix} Y_h \\ L \end{matrix}\right\} + \overset{*}{X}_{(h_1h_2)} \left\{\begin{matrix} \overset{*}{Y}_{h_1h_2} \\ L \end{matrix}\right\}. \tag{6.8}$$

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[6] The subscript k_1 being restricted to be less or equal to k_2, some obvious modification will have to be introduced in the summation convention involving vectors with double subscripts,

7. Lagrangian equations of motion and the reactions in hyperspace. Differentiating (4.6) the rate of change in the momentum vector is expressed as

$$\ddot{Y} = Y_k\ddot{q}^h + Y_{h_1h_2}\dot{q}^{h_1}\dot{q}^{h_2} \tag{7.1}$$

which shows that the vector $\ddot{Y}$ is in the subspace, adjoint to the osculating subspace of the second order of the variety at the point Q. In accordance with the Newtonian mechanics the motion of the generalized particle is expressible by the vector equation

$$\dot{Y} = \lambda \, . \, F, \tag{7.2}$$

where F is the generalized force vector, λ being a constant.

Substituting (7.1) in (7.2) and resolving it into its components we get the following sets of equations

$$\left\{ \begin{matrix} Y_k \\ X_k \end{matrix} \right\} \ddot{q}^k + \left\{ \begin{matrix} Y_{h_1h_2} \\ X_k \end{matrix} \right\} \dot{q}^{h_1}\dot{q}^{h_2} = \lambda . \left\{ \begin{matrix} F \\ X_k \end{matrix} \right\}, \tag{7.3}$$

$$\left\{ \begin{matrix} Y_{h_1h_2} \\ \overset{*}{X}_{k_1k_2} \end{matrix} \right\} \dot{q}^{h_1}\dot{q}^{h_2} = \lambda \left\{ \begin{matrix} F \\ \overset{*}{X}_{k_1k_2} \end{matrix} \right\}. \tag{7.4}$$

Defferentiating (5.5) and remembering the relations (5.8) it is easy to see that (7.3) is equivalent to the Lagrangian equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} = \lambda . \left\{ \begin{matrix} F \\ X_k \end{matrix} \right\}. \tag{7.5}$$

If now $\left\{ \begin{matrix} F \\ X_k \end{matrix} \right\}$ be given as a known function of the position coordinates q^k (which may involve $\dot{q}^k$ also), the r equations (7.3) or (7.5) are theoretically sufficient to determine q's as functions of t, when initial circumstances are taken into consideration.

The equations (7.4), on the other hand, determine the unknown components of the force vector F. These are of the nature of reactions in hyperspace and are expressed as homogeneous quadratic functions of the r velocity coordinates $\dot{q}^k$.

It may be noted that an explicit form of the generalized force vector[7] F is given by

$$F = Y_{(k)} \left\{ \begin{matrix} F \\ X_k \end{matrix} \right\} + \overset{*}{Y}_{(k_1k_2)} \left\{ \begin{matrix} F \\ \overset{*}{X}_{k_1k_2} \end{matrix} \right\} \tag{7.6}$$

also

$$\lambda \left\{ \begin{matrix} F \\ \overset{*}{X}_{(k_1k_2)} \end{matrix} \right\} = (\dot{q}^{k_1})^2, \ \lambda \left\{ \begin{matrix} F \\ \overset{*}{X}_{(k_1k_1)} \end{matrix} \right\} = 2\dot{q}^{k_1}\dot{q}^{k_2}. \tag{7.7}$$

8. Hamiltonian equations of motions. To derive Hamiltonian equations we start from the relation (5.6)

$$\dot{Y} = Y_{(l)}p_l.$$

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[7] We have not considered here what part this force vector will play in regard to a system of particles in ordnary space.

Differentiating this, we get

$$Y = Y_{(l)}\dot{p}_l + Y_{(l)h} p_l \dot{q}^h,$$

whence

$$\left\{\begin{matrix} \ddot{Y} \\ X_k \end{matrix}\right\} = \dot{p}_k + \left\{\begin{matrix} Y_{(l)h} \\ X_k \end{matrix}\right\} p_l \dot{q}^h. \tag{8.1}$$

Now

$$\left\{\begin{matrix} Y_{(l)h} \\ X_k \end{matrix}\right\} = -\left\{\begin{matrix} Y_{(l)} \\ X_{kh} \end{matrix}\right\} = \left\{\begin{matrix} Y_{(l)k} \\ X_h \end{matrix}\right\}$$

Consequently (8.1) reduces to

$$\left\{\begin{matrix} \ddot{Y} \\ X_k \end{matrix}\right\} = p_k + \left\{\begin{matrix} Y_{(l)k} \\ X_{(h)} \end{matrix}\right\} p_l p_h, \tag{8.2}$$

In view of (5.9), this is equevalent to

$$\left\{\begin{matrix} \ddot{Y} \\ X_k \end{matrix}\right\} = \dot{p}_k + \frac{\partial T'}{\partial q^k}\,. \tag{8.3}$$

Thus the Hamiltonian form corresponding to (7.5) is

$$\dot{p}_k + \frac{\partial T'}{\partial q^k} = \lambda \left\{\begin{matrix} F \\ X_k \end{matrix}\right\}, \tag{8.4}$$

with which is associated the system of equations (5.11)

$$\dot{q}^k = \frac{\partial T'}{\partial p_k}\,. \tag{8.5}$$

9. Motion on a varying variety. If the vector X in (3.1) giving the position of the particle at time t involve t explicity, as well as, implicity through the coordinates q^k we write

$$X = F(t, q^1, q^2, \ldots\ldots q^r). \tag{9.1}$$

At time $t+\partial t$, the vector changes into

$$X + \partial X = F(t+\partial t,\ q^1 + \dot{q}^1\partial t,\ q^2 + \dot{q}^2\partial t, \ldots\ .q^r + \dot{q}^r\partial t) \tag{9.2}$$

and the motion of the particle takes place on a varying variety of r dimensions. The velocity vector $\dot{X}$ is now of the form

$$\dot{X} = \frac{\partial X}{\partial t} + X_h \dot{q}^h, \tag{9.3}$$

whence the acceleration vector is given by

$$\ddot{X} = \frac{\partial^2 X}{\partial t^2} + 2\dot{q}^h \frac{\partial}{\partial t} X_h + X_{hk}\dot{q}^h \dot{q}^k + X_h \ddot{q}^h. \tag{9.4}$$

Introducing the symbol X_0 to denote $\partial X/\partial t$ and noting that $X_{h0} = X_{0h}$, we can write (9.3, 4) as

$$\dot{X} = X_\omega \dot{q}^\omega, \tag{9.5}$$

$$\ddot{X} = X_\omega \ddot{q}^\omega + X_{\omega_1\omega_2}\dot{q}^{\omega_1}\dot{q}^{\omega_2}, \tag{9.6}$$

where $\dot{q}^0 = 1$, $\ddot{q}^0 = 0$ and ω ranges over the values $(0, 1, 2, \ldots\ldots r)$.

The expression for the kinetic energy at time t is now given by

$$T = \tfrac{1}{2}\left\{\begin{matrix} X_{\varpi_1} \\ Y_{\varpi_2} \end{matrix}\right\}\dot{q}^{\varpi_1}\dot{q}^{\varpi_2}, \tag{9.7}$$

which is of the same form as (4.7), if the dummy indices run over the values $0, 1, 2, \ldots r$. All the results of §§ 5 and 6 may now be taken over almost unchanged in form with this extension in the range of values regarding dummy indices. The scalar determinents (5.1) and (6.1) are, of course, to be angmented to include vectors with zero subscripts.

10. Lagrange's equations. From (9.6), passing to the adjoint system of vectors, we have

$$\ddot{Y} = Y_{\varpi}\ddot{q}^{\varpi} + Y_{\varpi_1\varpi_2}\dot{q}^{\varpi_1}\dot{q}^{\varpi_2}. \tag{10.1}$$

The vector equation of motion being of the same form as in (7.2), corresponding to (7.3) and (7.4) we have now the following sets of equations:

$$\left\{\begin{matrix} Y_{\varpi_1} \\ X_k \end{matrix}\right\}\ddot{q}^{\varpi_1} + \left\{\begin{matrix} Y_{\varpi_1\varpi_2} \\ X_k \end{matrix}\right\}\dot{q}^{\varpi_1}\dot{q}^{\varpi_2} = \lambda\left\{\begin{matrix} F \\ X_k \end{matrix}\right\}, \tag{10.2}$$

$$\left\{\begin{matrix} Y^{\varpi_1} \\ X_0 \end{matrix}\right\}\ddot{q}^{\varpi_1} + \left\{\begin{matrix} Y_{\varpi_1\varpi_2} \\ X_0 \end{matrix}\right\}\dot{q}^{\varpi_1}\dot{q}^{\varpi_2} = \lambda\left\{\begin{matrix} F \\ X_0 \end{matrix}\right\}, \tag{10.2a}$$

$$\left\{\begin{matrix} Y_{\varpi_1\varpi_2} \\ \overset{*}{X}_{\varpi_3\varpi_4} \end{matrix}\right\}\dot{q}^{\varpi_1}\dot{q}^{\varpi_2} = \lambda\left\{\begin{matrix} F \\ \overset{*}{X}_{\varpi_3\varpi_4} \end{matrix}\right\}. \tag{10.3}$$

From the relations

$$\frac{\partial T}{\partial \dot{q}^k} = \left\{\begin{matrix} \dot{Y} \\ X_k \end{matrix}\right\} \text{ and } \frac{\partial T}{\partial q_k} = \left\{\begin{matrix} \dot{Y} \\ X_{\varpi k} \end{matrix}\right\}\dot{q}^{\varpi} \tag{10.4}$$

it is easy to see that the equations (10.2) reduce to the Lagrangian form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} = \lambda\left\{\begin{matrix} F \\ X_k \end{matrix}\right\}, \tag{10.5}$$

where T is given by (9.7). The equations (10.2) or (10.5) are sufficient to determine the r unknown quantities q^k. The equation (10.2a) will then determine $\left\{\begin{matrix} F \\ X_0 \end{matrix}\right\}$, the constraining forces, required to make $\ddot{q}^0 = 0$, $\dot{q}^0 = 1$. The unknown components of F are given by (10.3). These are not now expressible as homogeneous functions of the velocity coordinates $\dot{q}^k$ as in (7.4).

11. Hamiltonian equations. Since

$$\left.\begin{aligned} \dot{X} &= X_{(\varpi)}p_{\varpi}, \\ \dot{Y} &= Y_{(\varpi)}p_{\varpi}, \end{aligned}\right\} \tag{11.1}$$

where

$$p_{\varpi} = \left\{\begin{matrix} \dot{Y} \\ X_{\varpi} \end{matrix}\right\} = \left\{\begin{matrix} \dot{X} \\ Y_{\varpi} \end{matrix}\right\}, \tag{11.2}$$

the kinetic energy is expressible in the alternative form

$$T' = \tfrac{1}{2}\left\{ \begin{matrix} X_{(\varpi_1)} \\ Y_{(\varpi_2)} \end{matrix} \right\} p_{\varpi_1} p_{\varpi_2}. \tag{11.3}$$

From (11.1) we have by differentiation

$$\ddot{Y} = Y_{(\varpi)} \dot{p}_{\varpi} + Y_{(\varpi)\varpi_1} p_{\varpi} \dot{q}^{\varpi_1},$$

whence

$$\left\{ \begin{matrix} \ddot{Y} \\ X_k \end{matrix} \right\} = \left\{ \begin{matrix} Y_{(\varpi)} \\ X_k \end{matrix} \right\} \dot{p}_{\varpi} + \left\{ \begin{matrix} Y_{(\varpi)\varpi_1} \\ X_k \end{matrix} \right\} p_{\varpi} \dot{q}^{\varpi_1}$$

$$= \dot{p}_k + \left\{ \begin{matrix} Y_{(\varpi)k} \\ X_{\varpi_1} \end{matrix} \right\} p_{\varpi} \dot{q}^{\varpi_1}$$

$$= \dot{p}_k + \left\{ \begin{matrix} Y_{(\varpi)k} \\ X_{(\varpi_1)} \end{matrix} \right\} p_{\varpi} \dot{p}_{\varpi_1}. \tag{11.4}$$

Thus the Hamiltonian form corresponding to (10.5) is

$$\dot{p}_k + \frac{\partial T'}{\partial q^k} = \lambda . \left\{ \begin{matrix} F \\ X_k \end{matrix} \right\}, \tag{11.5}$$

where T' is given by (11.3). It may be noted that both T and T' are now expressible also in the form

$$\tfrac{1}{2}(p_0 + p_1 \dot{q}^1 + p_2 \dot{q}^2 + \dots + p_r \dot{q}^r). \tag{11.6}$$

From (11.3) we have, further,

$$\frac{\partial T'}{\partial p_k} = \left\{ \begin{matrix} \dot{Y} \\ X_{(k)} \end{matrix} \right\} = \left\{ \begin{matrix} Y_{\varpi} \\ X_{(k)} \end{matrix} \right\} \dot{q}^{\varpi} = \dot{q}^k, \tag{11.7}$$

which combined with (11.5) gives the complete system of Hamiltonian equations. This method is in agreement with that adopted in § 8, but it differs from the usual method of arriving at the equations by means of an auxiliary function $(p_1 \dot{q}^1 + p_2 \dot{q}^2 + \dots + p_r \dot{q}^r - T)$. It may be noted that the expression (11.6) does not materially differ from this.

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ON SELF-RECIPROCAL FUNCTIONS AND A NEW TRANSFORM

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Introduction—C. V. H Rao (1918) evaluated the integral

$$\int_0^\infty x^{\rho-1} J_\mu(ax) J_\nu\left(\frac{b}{x}\right) dx$$

$$= \frac{a^{\nu-\rho} b^\nu \Gamma(\frac{1}{2}\mu + \frac{1}{2}\varrho - \frac{1}{2}\nu)}{2^{2\nu-\rho+1}\Gamma(\nu+1)\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\varrho+1)} {}_0F_3[\nu+1,\ \tfrac{1}{2}(\nu-\mu-\varrho)+1,\ \tfrac{1}{2}(\nu+\mu-\varrho)+1;\ a^2b^2/16]$$

$$+ \frac{a^\mu b^{\mu+\rho} \Gamma(\frac{1}{2}\nu - \frac{1}{2}\mu - \frac{1}{2}\varrho)}{2^{2\mu+\rho+1}\Gamma(\mu+1)\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\varrho+1)} {}_0F_3[\mu+1,\ \tfrac{1}{2}(\mu-\nu+\varrho)+1,\ \tfrac{1}{2}(\nu+\mu+\varrho)+1;\ a^2b^2/16]$$

with the conditions $a,\ b > 0$ and $-R(\mu+\frac{3}{2}) < R(\varrho) < R(\nu+\frac{3}{2})$ with $\varrho = 0$, we define

$$\varpi_{\mu,\nu}(xy) = (xy)^{\frac{1}{2}} \int_0^\infty J_\nu(T) J_\mu\left(\frac{xy}{T}\right)\frac{dT}{T}\ .$$

It is easy to observe that

(*i*) $\varpi_{\mu,\nu}(z) = \varpi_{\nu,\mu}(z)$

(*ii*) $\varpi_{\nu-1,\nu}(z) = J_{2\nu-1}(2z^{\frac{1}{2}})$

(*iii*) $\varpi_{\mu,\nu}(z) = O(1)$, when z is small.

(*iv*) $\varpi_{\mu,\nu}(z) = O(z^{-\frac{1}{4}})$, when z is large.

Watson (1932) noticed that $\varpi_{\mu,}(xy)$ could be regarded as a kernel which gave rise to a transform *at any rate formally.*

In this paper, we shall investigate in detail the conditions under which the kernel may give rise to a transform. We shall also investigate the necessary and sufficient conditions and the conditions in general under which $\varphi(p)$ or $f(t)$ connected by the relation,

$$\varphi(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty (pt)^s K_m(pt) t^n f(t) dt.$$

are self-reciprocal under this transform.

The advantage of choosing this relation is

(*i*) if $s = \frac{1}{2}$, $m = \pm\frac{1}{2}$ and $n = 0$, this reduces to Laplace Transform,

(*ii*) if $s = \frac{1}{2}$ and $n = 0$ this reduces to Meijer s transform.

We have also investigated the necessary and sufficient conditions under which a function shall be self-reciprocal in this transform.

We have obtained certain other theorems connected with this transform. We have also laid down the rules for construction of functions self-reciprocal in the Hankel transform when functions self-reciprocal in this transform are known and vice-versa.

In art. 1., we give certain examples of functions which are either self-reciprocal or skew-reciprocal under this transform. We shall give the proof of one result only; the other results can be obtained in a similar manner and their proofs will come out in due course in a separate paper (Mitra & Dinesh Chandra). Some of the examples follow from the theorems which we have proved in this paper.

In art. 2., We prove some lemmas which may be regarded as extensions of Lerch's theorem (Lerch, 1903).

We shall say that when $f(x)$ is self-reciprocal in this transform, *viz.*, when

$$f(y)=\int_0^\infty f(x)\,\varpi_{\mu,\nu}(xy)\,dx,$$

$f(x)$ belongs to $R_{\mu,\nu}$.

1. The following functions are

(*a*) *Self-reciprocal*

(*i*) $f(y) = K_n(y)\,y^{\nu+\mu+\frac{1}{2}}$, $\nu = \mu+2n$; $\mu \geq -\frac{1}{2}$, $n \geq 0$,

(*ii*) $f(y) = y^{\frac{1}{2}}I_{-\frac{1}{4}}(\frac{1}{2}y)K_{-\frac{1}{4}}(\frac{1}{2}y)$, $\mu = \nu = -\frac{1}{4}$,

(*iii*) $f(y) = \frac{1}{2}\pi[I_o(y)-L_o(y)]$, $\mu=\nu=\frac{1}{2}$,

(*iv*) $f(y) = y^{\frac{1}{2}}J_o\left(\frac{y}{\sqrt{2}}\right)K_o\left(\frac{y}{\sqrt{2}}\right)$, $\mu=\nu=0$,

(*v*) $f(y) = \{J_o^2(\frac{1}{2}y)+y_o^2(\frac{1}{2}y)\}$, $\mu=\nu=-\frac{1}{2}$,

(*vi*) $f(y) = \displaystyle\int_0^\infty \frac{x^{m+1}J_{m+2}(x)\,dx}{(x^2+y)^{m+1}}$, $\mu=\nu=m+2$,

(*vii*) $f(y) = y^{\frac{1}{2}}\displaystyle\int_0^\infty (\sinh t)^{\frac{1}{2}}K_o(y\sinh t)\,dt$, $\mu=\nu=0$;

(*b*) *Skew-reciprocal*

(*i*) $f(y) = \frac{1}{2}\pi^2\,y^{\frac{1}{2}}[H_o(y)-Y_o(y)]-\pi/(4y^{\frac{1}{2}})$, $\mu=\nu=0$,

(*ii*) $f(y) = \frac{1}{2}\pi\,y^{\frac{1}{2}}[I_o(y)-L_o(y)]-1/(2y^{\frac{1}{2}})$, $\mu=\nu=0$,

(*iii*) $f(y) = y^{-\frac{1}{2}}[e^{-y}-\frac{1}{2}]$, $\mu=0$, $\nu=0$.

It is worth noticing that (*i*) in (*a*) degenerates into the familiar function $e^{-\frac{1}{4}y^2}y^{2\mu+3/2}$ which is $R_{2\mu+1}$, when we take $n=\frac{1}{2}$ and write $\frac{1}{2}y^2$ for y.

We know that

$$\int_0^\infty \frac{xJ_o(ax)dx}{(x^4+1)^{\frac{1}{2}}} = J_o\left(\frac{a}{\sqrt{2}}\right)K_o\left(\frac{a}{\sqrt{2}}\right)$$

By Hankel's Theorem (Hankel), we get

$$\int_0^\infty xJ_o(x\varrho)\,J_o(xk)\,K_o(xk)\,dk = 1/(\varrho^4+4k^4)^{\frac{1}{2}}.$$

Let

$$f(y) = y^{\frac{1}{2}}J_o\left(\frac{y}{\sqrt{2}}\right)K_o\left(\frac{y}{\sqrt{2}}\right)$$

Therefore the integral

$$\begin{aligned}\int_0^\infty f(y)\varpi_{0,0}(xy)\,dy &= \int_0^\infty\int_0^\infty (xy)^{\frac12} J_0(t)\,J_0\left(\frac{xy}{t}\right)f(y)\frac{dt}{t}dy\\ &= x^{\frac12}\int_0^\infty J_0(t)\frac{dt}{t}\int_0^\infty yJ_0\left(\frac{xy}{t}\right)J_0\left(\frac{y}{\sqrt2}\right)K_0\left(\frac{y}{\sqrt2}\right)dy\\ &= x^{\frac12}\int_0^\infty \frac{J_0(t)t\,dt}{(x^4+t^4)^{\frac12}}\\ &= x^{\frac12} J_0\left(\frac{x}{\sqrt2}\right)K_0\left(\frac{x}{\sqrt2}\right);\end{aligned}$$

Showing that $f(x)$ is self-reciprocal in this transform. The change in the order of integration is easily justified. The rest of the examples that we have given can be verified by the application of Hankel's theorem.

We have frequently inverted the repeated integral

$$\int_0^\infty dx\int_0^\infty f(x,\,y)\,dy$$

The sufficient conditions are

(a) (i) $f(x,\,y)$ is a continuous function of $(x,\,y)$ in $x\geq 0$, $y\geq 0$,

(ii) $\int_0^\infty f(x,\,y)\,dx$, $\int_0^\infty f(x,\,y)\,dy$ converge uniformly in the arbitrary intervals $0\leq y\leq b$, $0\leq x\leq a$, respectively,

(iii) $\int_0^\infty dx\int_0^y f(x,\,y)\,dy$ converges uniformly in $y\geq 0$.

Then $\int_0^\infty dx\int_0^\infty f(x,\,y)\,dy$ and $\int_0^\infty dy\int_0^\infty f(x,\,y)\,dx$, exist and are equal.

(b) The x-integral and the y-integrals are absolutely convergent and one of the repeated integrals is absolutely convergent.

2. **Lerch's Theorem.** *If* $f(p)=\int_0^\infty e^{-pt}F(t)\,dt$, $p\geq p_0$ *is satisfied by a continuous function* $F(t)$, *there is no other continuous function which satisfies the equation.*

Lemma 1. *Let*

$$\int_0^\infty\int_0^\infty e^{-p_1x_1-p_2x_2}F(x_1,\,x_2)\,dx_1\,dx_2=0$$

where $F(x_1,\,x_2)$ *is a continuous function in* $(0,\,\infty)$ *of both the variables and* $p_1,\,p_2>0$, *then*

$$F(x_1,\,x_2)\equiv 0.$$

Proof. Let

$$\int_0^\infty e^{-p_1x_1}F(x_1,\,x_2)\,dx_1=\psi(x_2)$$

Since the integral converges uniformly, $\psi(x_2)$ is a continuous function of x_2 in $(0, X_2)$, therefore

$$\int_0^\infty e^{-p_2 x_2}\psi(x_2)\,dx_2 = 0.$$

By Lerch's theorem $\psi(x_2) \equiv 0.$

Therefore

$$\int_0^\infty e^{-p_1 x_1}F(x_1, x_2)\,dx_1 = 0.$$

Hence $F(x_1, x_2) \equiv 0$, again by Lerch's theorem.

Lemma II. *Let $G(t)$ be a continuous function of t in $(0, \infty)$ and $O(t^m)$ when t is small, m being greater than or equal to $\frac{1}{2}$. Let* $\int_0^\infty K_m(pt)\,G(t)\,dt = 0, \quad p > 0.$

Then will $G(t) \equiv 0.$

Proof. We shall consider the positive values of m, since $K_{-m}(pt) = K_m(pt)$. We also suppose that m is not half an odd integer. When m is half an odd integer, we shall suppose $m = n + \frac{1}{2}$, where n is a positive integer.

Hence $K_{n+\frac{1}{2}}(pt) = e^{-pt} \times \{\text{a polynomial of degree } n\} \times t^{-n-\frac{1}{2}}$

Therefore

$$\int_0^\infty K_{n+\frac{1}{2}}(pt)\,G(t)\,dt = \int_0^\infty e^{-pt}G(t)t^{-n-\frac{1}{2}} \times \begin{pmatrix}\text{a polynomial}\\ \text{of degree } n\end{pmatrix} dt,$$

Since $G(t)$ is continuous in $(0, \infty)$ and also $G(t)t^{-n-\frac{1}{2}} \times$ (a polynomial af degree n) is continuous, we have by Lerch's theorem

$$G(t)t^{-n-\frac{1}{2}} \times (\text{a polynomial of degree } n) \equiv 0$$

whence

$$G(t) \equiv 0.$$

We know that

$$K_n(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \cos n\pi\, W_{0,n}(2x), \text{ for all values of } n, \text{ and}$$

$$W_{k,m}(x) = \frac{e^{-\frac{1}{2}x}x^k}{\Gamma(\frac{1}{2}-k+m)} \int_0^\infty t^{-k-\frac{1}{2}+m}\left(1 + \frac{t}{x}\right)^{k-\frac{1}{2}+m} e^{-t}\,dt,$$

$R(\frac{1}{2}+m-k) > 0.$

Therefore we get

$$\int_0^\infty e^{-pt}t^{-m}G(t)dt \int_0^\infty u^{m-\frac{1}{2}}(u+2pt)^{m-\frac{1}{2}}e^{-u}\,du = 0.$$

Now u, p, t are positive quantities. We make use of the inequality that *

$$(u+pt)^{2m-1} \geq \{u(u+2pt)\}^{m-\frac{1}{2}}, \quad m \geq \tfrac{1}{2}$$

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* When $m \geq 0$ but less than $\frac{1}{2}$ we take $G(t) = 0(t^{m+1})$ when t is small. It is possible to take $G(t) = 0(t^{m+1})$ all throughout. We use the integral.

$$K_m(z) = \tfrac{1}{2}(\tfrac{1}{2}z)^m \int_0^\infty \exp(-t - z^2/4t)\,dt/t^{m+1}; \quad m \geq 0$$

The result can also be proved with the help of Operational Calculus.

We find that

(*i*) the u-integral is absolutely convergent,

(*ii*) the t-integral is absolutely convergent,

(*iii*) one of the repeated integrals is absolutely convergent.

Therefore a change in the order of integration is justified.

Hence

$$\int_0^\infty u^{m-\frac{1}{2}}e^{-u}\,du\int_0^\infty e^{-pt}t^{-m}(u+2pt)^{m-\frac{1}{2}}G(t)\,dt = 0.$$

Let

$$F(u) = e^{-\frac{1}{2}u}u^{m-\frac{1}{2}}\int_0^\infty e^{-pt}t^{-m}(u+2pt)^{m-\frac{1}{2}}G(t)\,dt.$$

Then $F(u)$ is continuous in $(0, u)$.

We therefore have

$$\int_0^\infty e^{-\frac{1}{2}u}F(u)du = 0,$$

whence

$$F(u)\equiv 0.$$

Therefore

$$\int_0^\infty e^{-\frac{1}{2}pt}e^{-\frac{1}{2}pt}t^{-m}(u+2pt)^{m-\frac{1}{2}}G(t)\,dt = 0.$$

Setting

$$K(t) = e^{-\frac{1}{2}pt}t^{-m}(u+2pt)^{m-\frac{1}{2}}G(t)$$

where $K(t)$ is continuous in $(0, t)$, we find that $K(t)\equiv 0$, or $G(t)\equiv 0$.

Lemma III. *Let*[*]

$$\int_0^\infty \frac{tf(t)\,dt}{(t^2+b)^m} = \int_0^\infty \frac{t\,F(t)\,dt}{(t^2+b)^m},\quad b>0$$

where $m>0$ *and* $f(t)$ *and* $F(t)$ *or more generally* $f(t)-F(t)$ *are continuous. Then*

$$f(t)\equiv F(t).$$

Proof. Since $\displaystyle\int_0^\infty \frac{tG(t)\,dt}{(t^2+b)^{m+1}}$ is uniformly convergent in b in (b_1, b_2), we can differentiate with respect to b. The process can be repeated n times.

Therefore we get

$$\int_0^\infty \frac{tG(t)\,dt}{(t^2+b)^{m+n+1}} = 0.$$

Now, let

$$t^2+b = b/z.$$

The integral reduces to

$$\int_0^\infty G[\sqrt{\{b(1-z)/z\}}]z^m z^{n-1}dz = 0,\quad (n = 1, 2, 3\ldots).$$

Let

$$\varphi(z) = G[\sqrt{\{b(1-z)/z\}}],$$

then

$$\varphi(1) = G(0),\quad \varphi(0) = \underset{z\to\infty}{Lt}\, G(z),$$

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[*] The Lemma is also true if we take $f(t)$ and $F(t)$ instead of $tf(t)$ and $tF(t)$.

so that $\varphi(z)$ is continuous in $0 \leq z \leq 1$, add we get

$$\int_0^1 \varphi(z) z^m z^{n-1} dz = 0, \quad (n = 1, 2, 3 \ldots).$$

Therefore, by the Lemma of Lerch's theorem,

$$\varphi(z) \equiv 0 \text{ or } G(t) \equiv 0.$$

We can put $b = p$, also and multiply both sides by p. On interpretation, we get

$$x^{m-1} \int_0^\infty t e^{-x^2 t} G(t) dt = 0. \quad (x > 0).$$

Hence by Lerch's theorem

$$G(t) \equiv 0.$$

Lemma IV. *Let $f(t)$ be continuous and absolutely integrable in $(0, \infty)$ and*

$$\int_0^\infty J_\nu(xt) f(t)\, dt = 0, \qquad \nu > -\tfrac{1}{2}$$

then will

$$f(t) \equiv 0.$$

Proof. Let us multiply both sides by $e^{-ax} x_\nu$ and integrate with respect to x between the limits zero and infinity. We get

$$\int_0^\infty e^{-ax} x_\nu\, dx \int_0^\infty J_\nu(xt) f(t)\, dt = 0.$$

Changing the order of integration which is easily justifiable, we get

$$\int_0^\infty \frac{f(t)\, t^\nu\, dt}{(a^2 + t^2)^{\nu + \frac{1}{2}}} = 0.$$

By proceeding as in lemma III, we can show that

$$f(t) \equiv 0.$$

3. Let

$$\varphi(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty (pt)^s K_m(pt)\, t^n f(t)\, dt, \quad n + s + m > -\tfrac{1}{2}$$

Theorem I. *Let $f(t)$ and $f(1/t)$ be continuous and absolutely integrable in $(0, \infty)$. The necessary and sufficient condition that $p^{n+\frac{1}{2}} \varphi(p)$ shall be $R_{n+s-m,\, n+s+m}$ (self-reciprocal under this transform) is that $f(t) \equiv f(1/t)$.*

Proof. The condition is sufficient.

Let $f(t) \equiv f(1/t)$.

Then

$$\int_0^\infty p^{n+\frac{1}{2}} \varpi_{n+s-m,\, n+s+m}(py) \varphi(p)\, dp$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty p^{n+s+\frac{1}{2}} \varpi_{n+s-m,\, n+s+m}(py)\, dp \int_0^\infty t^{n+s} K_m(pt)\, f(t)\, dt$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty t^{n+s} f(t)\, dt \int_0^\infty p^{n+s+\frac{1}{2}} \varpi_{n+s-m,\, n+s+m}(py) K_m(pt)\, dp,$$

Since the change in order of integration is permissible under the conditions stated. Thus, the integral

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty t^{n+s}f(t)\,dt\left\{\int_0^\infty p^{n+s+\frac{1}{2}}(py)^{\frac{1}{2}}K_m(pt)\,dp\int_0^\infty J_{n+s+m}(T)J_{n+s-m}\left(\frac{py}{T}\right)\frac{dT}{T}\right\}$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}y^{\frac{1}{2}}\int_0^\infty t^{n+s}f(t)\,dt\left\{\int_0^\infty J_{n+s+m}(T)\,\frac{dT}{T}\int_0^\infty p^{n+s+1}K_m(pt)J_{n+s-m}\left(\frac{py}{T}\right)dp\right\}$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}y^{n+\frac{1}{2}}\int_0^\infty\left(\frac{y}{t}\right)^{s}\left(\frac{1}{t}\right)^{n+2}K_m\left(\frac{y}{t}\right)f(t)\,dt$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}y^{n+\frac{1}{2}}\int_0^\infty(yt)^s t^n K_m(yt)f\left(\frac{1}{t}\right)dt.$$

Therefore
$$\int_0^\infty p^{n+\frac{1}{2}}\varpi_{n+s-m,\,n+s+m}(py)\,\varphi(p)\,dp=y^{n+\frac{1}{2}}\varphi(y).$$

We may note here that all the changes in the order of integration are permissible under the conditions stated before.

The condition is necessary.

Let $p^{n+\frac{1}{2}}\varphi(p)$ be self-reciprocal in this transform *i.e.*, let $p^{n+\frac{1}{2}}\varphi(p)$ be $R_{n+s-m,\,n+s+m}$.

Then
$$\int_0^\infty p^{n+\frac{1}{2}}\varpi_{n+s-m,\,n+s+m}(py)\varphi(p)dp=y^{n+\frac{1}{2}}\varphi(y)$$

or
$$y^{n+\frac{1}{2}}\varphi(y)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty p^{n+s+\frac{1}{2}}\varpi_{n+s-m,\,n+s+m}(py)dp\int_0^\infty t^{n+s}K_m(pt)\,f(t)\,dt$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty t^{n+s}f(t)dt\int_0^\infty p^{n+s+\frac{1}{2}}K_m(pt)\varpi_{n+s-m,\,n+s+m}(py)\,dp$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}y^{n+\frac{1}{2}}\int_0^\infty(yt)^sK_m(yt)\,t^nf\left(\frac{1}{t}\right)dt;$$

after reductions and changes in the order of integration which are easily justifiable.

Therefore
$$\int_0^\infty t^{n+s}K_m(yt)\left[f(t)-f\left(\frac{1}{t}\right)\right]=0$$

Hence, by the second lemma,

$$f(t)\equiv f\left(\frac{1}{t}\right)$$ [*]

Theorem II *Let $f(t)$ be continuous and absolutely integrable in $(0,\infty)$. The necessary and sufficient condition that $f(t)$ shall be self-reciprocal is that*

$$\varphi_1(p)=p^{-2n-2}\varphi_1\left(\frac{1}{p}\right);\quad n+s\pm m>-\tfrac{1}{2},$$

---

[*] We note that $t^{n+s}[f(t)-f(1/t)]$ should be $O(t^n)$; $m\geq\frac{1}{2}$ when t is small. It is sufficient if we suppose that the order in $O(t^{m+1})$ when $m\geq 0$.

where

$$\varphi_1(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty (pt)^s K_m(pt)\, t^{n+\frac{1}{2}} f(t)\, dt.$$

Proof. The condition is necessary.

Let $f(t)$ be self-reciprocal in this transform *i.e.*, let $f(t)$ be $R_{n+s-m,\, n+s+m}$.

Then

$$\varphi_1(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty (pt)^s t^{n+\frac{1}{2}} K_m(pt)\, dt \int_0^\infty f(y) \varpi_{n+s-m,\, n+s+m}(yt)\, dt$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty p^s f(y) dy \int_0^\infty t^{n+s+\frac{1}{2}} K_m(pt) \varpi_{n+s-m,\, n+s+m}(yt)\, dt$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} p^s \int_0^\infty y^{\frac{1}{2}} f(y)\, dy \left\{ \int_0^\infty J_{n+s+m}(T) \frac{dT}{T} \int_0^\infty t^{n+s+1} J_{n+s+m}\left(\frac{yt}{T}\right) K_m(pt)\, dt \right\}$$

$$= p^{-2n-2} \varphi_1\left(\frac{1}{p}\right)$$

after a little simplification.

We have here changed the order of integration twice; both being easily justifiable under the condition stated.

The condition is sufficient.

Let

$$\varphi_1(p) = p^{-2n-2} \varphi_1\left(\frac{1}{p}\right).$$

Then

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \left(\frac{t}{p}\right)^s K_m\left(\frac{t}{p}\right) t^{n+\frac{1}{2}} f(t)\, dt$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} p^{2n+2} \int_0^\infty (pt)^s K_m(pt)\, t^{n+\frac{1}{2}} f(t)\, dt. \tag{A}$$

Now,

$$\int_0^\infty (pt)^s K_m(pt) t^{n+\frac{1}{2}} \varpi_{n+s-m,\, n+s+m}(yt) dt$$

$$= \int_0^\infty (pt)^s K_m(pt) t^{n+\frac{1}{2}} (yt)^{\frac{1}{2}} dt \int_0^\infty J_{n+s+m}(T) J_{n+s-m}\left(\frac{yt}{T}\right) \frac{dT}{T}$$

$$= p^{-2n-2} \left(\frac{y}{p}\right)^s y^{n+\frac{1}{2}} K_m\left(\frac{y}{p}\right)$$

after slight reduction and an easily justifiable change in the order of integration.

Hence we get from (A)

$$\int_0^\infty (pt)^s K_m(pt(t^{n+\frac{1}{2}} f(t)\, dt$$

$$= \int_0^\infty f(t) dt \int_0^\infty (yp)^s K_m(yp) y^{n+\frac{1}{2}} \varpi_{n+s-m,\, n+s+m}(yt)\, dy$$

$$= \int_0^\infty (yp)^s K_m(yp) y^{n+\frac{1}{2}} F(y)\, dy,$$

where

$$y^{\frac{1}{2}}F(y) = \int_0^\infty \frac{(yt)^{\frac{1}{2}} f(t)}{t^{\frac{1}{2}}}\, \varpi_{n+s-m,\, n+s+m}(yt)\, dt. \tag{B}$$

Therefore **

$$\int_0^\infty (pt)^s K_m(pt) t^{n+\frac{1}{2}}[t^{\frac{1}{2}}f(t) - t^{\frac{1}{2}}F(t)]dt = 0.$$

Let

$$G(t) = t^{\frac{1}{2}}[f(t) - F(t)].$$

Then, since (*i*) $f(t)$ is a *continuous function* of t and (*ii*) (B) is *uniformly convergent*, $y^{\frac{1}{2}}F(y)$ is or continuous function of y. Therefore by our second lemma $F(t) \equiv f(t)$, showing that $f(t)$ is self-reciprocal in this transform.

Corollary. *If we take* $s = \frac{1}{2}$, $m = \frac{1}{2}$, *we get*

$$\varphi(p) = \int_0^\infty e^{-pt} f(t) t^n\, dt$$

and

$$\varpi_{n+s-m,\, n+s+m}(xy) = \varpi_{n,\, n+1}(xy) = J_{2n+1}(2(xy)^{\frac{1}{2}}).$$

Tricomi's results (Tricomi, 1935) are at once deducible. We shall now illustrate our theorems by suitable examples.

Examples. (*i*) Let, $f(t) = t/(1+t^2)$, *i.e.*, $f(t) = f(1/t)$. For $s = 0$, $m = 0$, $n = 1$, we prove that

$$p^{3/2}\{\pi/(2p) - \tfrac{1}{4}\pi^2[H_0(p) - Y_0(p)]\} \text{ is } R_{1,\,1}.$$

(*ii*) If $f(t) = t^{\frac{1}{2}}/(1+t^2)^{\frac{1}{2}}$, $s = 0$, $m = 0$, $n = -\frac{1}{2}$,

then

$$\tfrac{1}{8}\pi^2\{J_0^2(\tfrac{1}{2}p) + Y_0^2(\tfrac{1}{2}p)\} \quad \text{is} \quad R_{-\frac{1}{2},\,-\frac{1}{2}}.$$

(*iii*) If $f(t) = t^2/(t^2+2)(2t^2+1)$, $s = 0$, $m = 0$, $n = 2$,

then

$$p^{5/2}\left\{\frac{6\sqrt{2}}{\pi p} - 8[H_0(\sqrt{2}p) - Y_0(\sqrt{2}p)] + H_0\left(\frac{p}{\sqrt{2}}\right) - Y_0\left(\frac{p}{\sqrt{2}}\right)\right\} \text{ is } R_{2,\,2}.$$

(*iv*) If $f(t) = t^{\frac{1}{2}}/(1+t^2)^{\frac{1}{2}}$; $s = -n$,

$$p^{\frac{1}{2}}F(p) = \int_0^\infty \frac{(pt)^{\frac{1}{2}}}{(1+t^2)^{\frac{1}{2}}} K_m(pt)\, dt \text{ is } R_{-m,\, m} \quad (0 < m < 1)$$

when

$$m = \tfrac{1}{2}, \quad F(p) = (\pi^3/8p)^{\frac{1}{2}}[H_0(p) - Y_0(p)],$$

and we fall back upon the known result

$$\int_0^\infty [H_0(p) - Y_0(p)] J_0(2(py)^{\frac{1}{2}})\, dp = [H_0(y) - Y_0(y)].$$

(*v*) If $f(t) = t^{\frac{1}{2}\nu}/(1+t)^{\frac{1}{2}\nu}$, $n+s = \frac{1}{4}\nu$,

$$\varphi(p) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} p^{\frac{1}{4}\nu - n} \int_0^\infty \frac{t^{\frac{1}{2}\nu} K_m(pt)\, dt}{(1+t)^{\frac{1}{2}\nu}}; \quad \pm m + \tfrac{1}{2}\nu > -1 \quad (0 > m > 1)$$

i.e.,

$$p^{n+\frac{1}{2}}\varphi(p) \quad \text{is} \quad R_{\frac{1}{4}\nu - m,\, \frac{1}{4}\nu + m}.$$

In particular, if $m = \frac{1}{2}$,

** We note that $t^{n+s+\frac{1}{2}}[f(t) - F(t)]$ should be $O(t^m)$ when t is small, $m \geq \frac{1}{2}$. We can take the order to be $O(t^{m+1})$ when $m \geq 0$.

$$\varphi(p) = 2^{\frac12\nu}\Gamma\{(\nu+1)/2\}e^{\frac12 p}p^{\frac12\nu-n-1}D_{-\nu}(2p)^{\frac12}.$$

and we get the known result

$$\int_0^\infty x^\nu J_\nu(xz)e^{\frac14 x^2}D_{-2\nu}(x)\,dx = z^{\nu-1}e^{\frac14 z^2}D_{-2\nu}(z)\,; \quad (\nu > -\tfrac12).$$

(*vi*) If

$$f(t) = \frac{t^{\frac12\nu}}{(1+t)^\nu}, \qquad n+s = \tfrac12(1-\nu)$$

$$\varphi(p) = \left(\frac{2}{\pi}\right)^{\frac12}\int_0^\infty p^{\frac12(1-\nu)-n}\frac{t^{\frac12}K_m(pt)\,dt}{(1+t)^\nu}$$

i.e., $p^{n+\frac12}\varphi(p)$ is $R_{\frac12(1-\nu)-m,\ \frac12(1-\nu)+m}$.

In particular, if $m = \frac12$, then

$$\varphi(p) = p^{-n-1}e^{\frac12 p}W_{-\frac12\nu,\ \frac12(1-\nu)}(p)$$

and we have

$$\int_0^\infty p^{-\frac12}e^{\frac12 p}W_{-\frac12\nu,\ \frac12(1-\nu)}(p)J_{1-\nu}(2(py)^{\frac12})\,dp = y^{-\frac12}e^{\frac12 y}W_{-\frac12\nu,\ \frac12(1-\nu)}(y) \quad (-\tfrac12 < R(\nu) < 2).$$

(*vii*) If

$$f(t) = \frac{t^{-\frac12\mu-\frac12\nu+\frac14}}{(1+t)^{-\mu-\nu+\frac12}}; \quad n+s = \frac{3\nu}{2}-\tfrac12\mu-\tfrac14,$$

$$\varphi(p) = \left(\frac{2}{\pi}\right)^{\frac12}p^{3\nu/2-\frac12\mu-\frac14-n}\int_0^\infty\frac{t^{\nu-\mu}K_m(pt)\,dt}{(1+t)^{-\mu-\nu+\frac12}}$$

i.e., $p^{n+\frac12}\varphi(p)$ is $R_{3\nu/2-\frac12\mu-\frac14-m,\ 3\nu/2-\frac12\mu-\frac14+m}$.

In particular, if $m = \frac12$, then

$$\varphi(p) = \Gamma(\nu-\mu+\tfrac12)p^{\frac12\nu-\frac12\mu-n-5/4}e^{\frac12 p}W_{\mu,\nu}(p).$$

Therefore

$$\int_0^\infty p^{\frac12(\nu-\mu-3/2)}e^{\frac12 p}W_{\mu,\nu}(p)J_{3\nu-\mu-\frac12}(2(py)^{\frac12})dp = y^{\frac12(\nu-\mu-3/2)}e^{\frac12 y}W_{\mu,\nu}(y),$$

$$R(6\nu-2\mu) > -1; \quad R(2\nu-2\mu) > -1; \quad R(\mu+\nu) < 1.$$

In particular, if we take $\mu = -\frac12\nu$ and write $\frac12(1-\nu)$ for ν, then $p^{\frac12(1-\nu)}s(\nu, p)$ is $R_{1-\nu}$; $(\nu < 1)$

where $s(\nu, p) = p^{\frac12\nu-1}e^{\frac12 p}W_{-\frac12\nu,\ \frac12(1-\nu)}(p)$.

(*viii*) If $f(t) = \Gamma(\alpha)[t^{\frac12\alpha}/(1+t)^\alpha]$, $m = \frac12$, $s = \frac12$, $n = \nu-1$, $(\nu > 0)$,

then $\varphi(p) = (1/p^{\nu+\frac12\alpha})\,E(\alpha,\ \nu+\frac12\alpha :: p)$;

i.e.,

$$\int_0^\infty p^{-\frac12(\alpha+1)}E(\alpha,\ \nu+\tfrac12\alpha :: p)\varpi_{\nu-1,\nu}(yp)\,dp = y^{-\frac12(\alpha+1)}E(\alpha,\ \nu+\tfrac12\alpha :: y)$$

a result which has been proved independently by N. N. Bose. Here E stands for MacRobert's Function, *viz.*,

$$E(\alpha,\ \beta :: x) = \Gamma(\alpha)\Gamma(\beta)x^{\frac12(\alpha+\beta-1)}e^{\frac12 x}W_{\frac12-\frac12(\alpha+\beta),\ \frac12(\beta-\alpha)}(x).$$

Theorem III. *Let $f(t)$ be continuous and absolutely integrable in $(0, \infty)$. Further let*

$$\psi(p)=\int_0^\infty \frac{t^{\nu+\frac12}f(t)\,dt}{(t^2+p^2)^{\nu+\frac12}}, \qquad (\nu \geq \tfrac12).$$

If $f(t)$ *is* R_ν, *then provided* $p^{2\nu}\psi(p)=\psi(1/p)$;

$$\int_0^\infty t^{\nu-\frac12}J_{2\nu-1}(2(xt)^{\frac12})\,\psi(t)\,dt=\int_0^\infty t^{\nu-\frac12}(xt)^{\frac12}J_\nu(xt)\psi(t)\,dt. \qquad (B_1)$$

We have

$$\psi(p)=\int_0^\infty \frac{t^{\nu+\frac12}}{(t^2+p^2)^{\nu+\frac12}}dt\int_0^\infty (yt)^{\frac12}J_\nu(yt)f(y)\,dy$$

$$=\int_0^\infty y^{\frac12}f(y)\,dy\int_0^\infty \frac{t^{\nu+1}J_\nu(yt)\,dt}{(t^2+p^2)^{\nu+\frac12}}$$

$$=\frac{\pi^{\frac12}}{2^\nu\Gamma(\nu+\frac12)}\int_0^\infty e^{-yp}y^{\nu-\frac12}f(y)\,dy.$$

Therefore

$$\int_0^\infty p^{\nu-\frac12}\psi(p)\,\varpi_{\nu-1,\,\nu}(px)dp=\frac{\pi^{\frac12}}{2^\nu\Gamma(\nu+\frac12)}\int_0^\infty y^{\nu-\frac12}f(y)\,dy\int_0 e^{-yp}\varpi_{\nu-1,\,\nu}(xp)p^{\nu-\frac12}dp$$

$$=\frac{(\pi x)^{\frac12}}{2^\nu\Gamma(\nu+\frac12)}\int_0^\infty y^{\nu-\frac12}f(y)\,dy\left\{\int_0^\infty e^{-yp}p^\nu\,dp\int_0^\infty J_\nu(T)\,J_{\nu-1}\left(\frac{xp}{T}\right)\frac{dT}{T}\right.$$

$$=\frac{(\pi x)^{\frac12}}{2^\nu\Gamma(\nu+\frac12)}\int_0^\infty y^{\nu-\frac12}f(y)\,dy\left\{\int_0^\infty J_\nu(Tx)\,\frac{dT}{T}\int_0^\infty e^{-yp}p^\nu J_{\nu-1}\left(\frac{p}{T}\right)dp\right\}$$

$$=x^{\frac12}\int_0^\infty \frac{J_\nu(Tx)\,dT}{T^\nu}\int_0^\infty \frac{y^{\nu+\frac12}f(y)\,dy}{(y^2+1/T^2)^{\nu+\frac12}}$$

$$=x^{\frac12}\int_0^\infty \frac{J_\nu(Tx)}{T^\nu}\psi\left(\frac{1}{T}\right)dT$$

$$=\int_0^\infty T^{\nu-\frac12}(Tx)^{\frac12}J_\nu(Tx)\psi(T)\,dT.$$

Since $\varpi_{\nu-1,\,\nu}(xp)=J_{2\nu-1}(2(xp)^{\frac12})$, the relation (B_1) follows.

The change in the orders of integrations are easily justified.

Again, since

$$\psi(p)=\frac{\pi^{\frac12}}{2^\nu\Gamma(\nu+\frac12)}\int_0^\infty e^{-yp}y^{\nu-\frac12}f(y)\,dy,$$

$$\int_0^\infty \psi(p)(px)^{\frac12}J_{\nu-1}(px)p^{\nu-\frac12}\,dp=\frac{(\pi x)^{\frac12}}{2^\nu\Gamma(\nu+\frac12)}\int_0^\infty y^{\nu-\frac12}f(y)\,dy\int_0^\infty e^{-yp}p^\nu J_{\nu-1}(xp)\,dp$$

$$=x^{\nu-\frac12}\int_0^\infty \frac{y^{\nu+\frac12}f(y)\,dy}{(x^2+y^2)^{\nu+\frac12}}$$

$$=x^{\nu-\frac12}\psi(x)$$

which shows that $x^{\nu-\frac12}\psi(x)$ is $R_{\nu-1}$.

As a corollary, we have, by taking $\nu = \frac{1}{2}$

$$\int_0^\infty \psi(p)\, J_0(2(px)^{\frac{1}{2}})\, dp = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \psi(t) \sin xt\, dt$$

provided

$$\psi(p) = \frac{1}{p}\psi\left(\frac{1}{p}\right).$$

Theorem IV. *Let* $x^{s-\frac{1}{2}}f(x)$ *be continuous and absolutely integrable in* $(0, \infty)$ *and let*

$$\varphi_2(p) = \int_0^\infty x^{s-\frac{1}{2}} J_\nu(xp) f(x)\, dx.$$

Then if,

$$f(1/x) = x^{2s} f(x); \qquad (s \geq 0),$$

$$\int_0^\infty \varphi_2(p) J_{2\nu}(2(py)^{\frac{1}{2}})\, dp = \varphi_2(y); \qquad (\nu \geq 0).$$

Proof.

$$\begin{aligned}
\int_0^\infty \varphi_2(p) J_{2\nu}(2(px)^{\frac{1}{2}})\, dp &= \int_0^\infty J_{2\nu}(2(py)^{\frac{1}{2}})\, dp \int_0^\infty x^{s-\frac{1}{2}} J_\nu(xp) f(x)\, dx \\
&= \int_0^\infty x^{s-\frac{1}{2}} f(x)\, dx \int_0^\infty J_\nu(xp) J_{2\nu}(2(py)^{\frac{1}{2}}) dp \\
&= \int_0^\infty x^{s-3/2} f(x)\, dx \int_0^\infty J_\nu(p) J_{2\nu}(2(py/x)^{\frac{1}{2}}) dp \\
&= \int_0^\infty x^{s-3/2} J_\nu(y/x) f(x)\, dx \\
&= \int_0^\infty x^{-s-\frac{1}{2}} f(1/x) J_\nu(xy)\, dx \\
&= \int_0^\infty x^{s-\frac{1}{2}} f(x) J_\nu(xy)\, dx \\
&= \varphi_2(y),
\end{aligned}$$

since,

$$f(1/x) = x^{2s} f(x)$$

Example. Let $s = 0$ and $f(x) = x^{\frac{1}{2}}/(1+x^2)^{\frac{1}{2}}$,

then

$$\varphi(p) = I_{\frac{1}{2}\nu}(\tfrac{1}{2}p) K_{\frac{1}{2}\nu}(\tfrac{1}{2}p).$$

Therefore

$$\int_0^\infty I_{\frac{1}{2}\nu}(\tfrac{1}{2}p) K_{\frac{1}{2}\nu}(\tfrac{1}{2}p) J_{2\nu}(2(xp)^{\frac{1}{2}})\, dp = I_{\frac{1}{2}\nu}(\tfrac{1}{2}x) K_{\frac{1}{2}\nu}(\tfrac{1}{2}x), \qquad (\nu > -\tfrac{1}{2}).$$

The particular case when $\nu = 0$ has been investigated by Mitra and Brij Mohan (1939). The general case has been investigated by Erdelyi (1938).

Theorem V. *Conversely, let* $s \geq 0$ *and* $x^{s-\frac{1}{2}}f(x)$ *and* $x^{s-3/2}f(x)$ *be continuous and absolutely integrable in* $(0, \infty)$.

Further, let

$$\varphi(y) = \int_0^\infty \varphi(p) J_{2\nu}(2(py)^{\frac{1}{2}})\, dp, \qquad (\nu \geq 0).$$

Then will $$x^{2s}f(x) = f(1/x).$$

We have
$$\begin{aligned}
\varphi(y) &= \int_0^\infty J_{2\nu}(2(py)^{\frac{1}{2}})\,dp \int_0^\infty x^{s-\frac{1}{2}}J_\nu(xp)f(x)\,dx \\
&= \int_0^\infty x^{s-\frac{1}{2}}f(x)\,dx \int_0^\infty J_\nu(xp)J_{2\nu}(2(py)^{\frac{1}{2}})\,dp \\
&= \int_0^\infty x^{s-3/2}J_\nu(y/x)f(x)\,dx \\
&= \int_0^\infty x^{-s-\frac{1}{2}}J_\nu(xy)f(1/x)\,dx && (C) \\
&= \int_0^\infty x^{s-\frac{1}{2}}J_\nu(xy)f(x)\,dx \qquad \text{(by hypothesis).} && (D)
\end{aligned}$$

Both (C) and (D) being uniformly convergent, we write
$$\int_0^\infty J_\nu(xy)[x^{s-\frac{1}{2}}f(x) - x^{-s-\frac{1}{2}}f(1/x)]dx = 0. \tag{E}$$

Since (E) is uniformly convergent, we multiply both sides by $y^{\nu+1}/(a^2+y^2)^{\nu+3/2}$ and integrate with respect to y between zero and infinity. On changing the order of intgration, which is permissible, we get
$$\int_0^\infty e^{-ax}x^\nu\{x^{s-\frac{1}{2}}f(x) - x^{-s-\frac{1}{2}}f(1/x)\}\,dx = 0.$$

Hence, by Lerch's theorem,
$$x^{2s}f(x) = f(1/x).$$

4. We shall now prove the necessary and sufficient condition that a function shall be $R_{\mu,\nu}$.

*Let $f(x)$ be continuous and absolutely integrable in $(0, \infty)$. The necessary and sufficient conditions** *that $f(x)$ shall be $R_{\mu,\nu}$ is that*
$$\int_0^\infty (xt)^{\frac{1}{2}}J_\nu(xt)f(t)\,dt = x^{-3/2}\int_0^\infty y^{\frac{1}{2}}f(y)J_\mu(y/x)\,dy, \qquad (\mu, \nu > -\tfrac{1}{2}).$$

If $\mu = \nu$, $$\varphi(x) = 1/x\,\varphi(1/x),$$
where
$$\varphi(x) = \int_0^\infty (xt)^{\frac{1}{2}}J_\nu(xt)f(t)\,dt.$$

The condition is sufficient.

---

* By Hankel's theorem, $f(x) = \int_0^\infty (xt)^{\frac{1}{2}}J_\nu(xt)\phi(t)dt$. If we choose $\phi(x) = \frac{1}{x}\phi(1/x)$, then we can find $f(x)$ which shall be $R_{\nu,\nu}$

We have
$$\int_0^\infty f(y)\varpi_{\mu,\nu}(xy)\,dy = \int_0^\infty f(y)\,dy\int_0^\infty (xy)^{\frac12}J_\nu(t)J_\mu\left(\frac{xy}{t}\right)\frac{dt}{t}$$
$$= \int_0^\infty f(y)\,dy\int_0^\infty (xy)^{\frac12}J_\nu(xt)J_\mu\left(\frac{y}{t}\right)\frac{dt}{t}$$
$$= \int_0 J_\nu(xt)\frac{dt}{t^{\frac12}}\int_0^\infty\left(\frac{xy}{t}\right)^{\frac12}J_\mu\left(\frac{y}{t}\right)f(y)\,dy$$
$$= \int_0^\infty (xt)^{\frac12}J_\nu(xt)\,dt\int_0^\infty (yt)^{\frac12}J_\nu(yt)f(y)$$
$$= f(x),$$
by Hankel's theorem.

The various changes in the order of integration can be easily justified. §§

The condition is necessary.

By Hankel's Theorem, **
$$f(x) = \int_0^\infty (xt)^{\frac12}J_\nu(xt)\,dt\int_0^\infty (yt)^{\frac12}J_\nu(yt)f(y)\,dy.$$

Also
$$f(x) = \int_0^\infty f(y)\varpi_{\mu,\nu}(xy)\,dy$$
$$= \int_0^\infty f(y)dy\int_0^\infty (xy)^{\frac12}J_\nu(t)J_\mu\left(\frac{xy}{t}\right)\frac{dt}{t}$$
$$= \int_0^\infty f(y)dy\int_0^\infty (xy)^{\frac12}J_\nu(xt)J_\mu\left(\frac{y}{t}\right)\frac{dt}{t}$$
$$= \int_0^\infty (xt)^{\frac12}J_\nu(xt)\,dt\int_0^\infty \frac{1}{t}\left(\frac{y}{t}\right)^{\frac12}J_\mu\left(\frac{y}{t}\right)f(y)\,dy$$

Let
$$\int_0^\infty (yt)^{\frac12}J_\nu(yt)f(y)\,dy = g(t)$$
and
$$\int_0^\infty\left(\frac{y}{t}\right)^{\frac12}J_\mu\left(\frac{y}{t}\right)f(y)\,dy = h(t).$$

Then $g(t)$ and $h(t)$ are continuous functions of t in $(0, T)$. Hence
$$\int_0^\infty (xt)^{\frac12}J_\nu(xt)[g(t)-t^{-1}h(t)]dt = 0.$$

Then provided $g(t)t^{\frac12}$ and $h(t)t^{-\frac12}$ are continuous and absolutely integrable in $(0, \infty)$ we can prove by lemma IV, that

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§§ Vide Watson, *Bessel Functions*

** $f(x)$ is continuous and absolutely integrable in $(0, \infty)$

$$g(t) = t^{-1}h(t)$$

and the condition follows.

5. We shall now obtain certain solutions of the problem.

Let $\tau(s)$ be the Mellin transform (Mellin, 1896) of $f(x)$. Let $f(t)$ be continuous and $t^{c-1}f(t)$ belong to $L(0, \infty)$, then, if

$$\tau(s) = \int_0^\infty x^{s-1}f(x)\,dx; \qquad s = c+it$$

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tau(s)x^{-s}ds.$$

Also, let $\tau(c+it)$ belong to $L(-\infty, \infty)$. Then, if

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tau(s)x^{-s}ds$$

$$\tau(x) = \int_0^\infty x^{s-1}f(x)\,dx.$$

Here

$$f(x) = \int_0^\infty \varpi_{\mu,\nu}(xy)f(y)\,dy.$$

Hence

$$\tau(s) = \int_0^\infty x^{s-1}\,dx\int_0^\infty \varpi_{\mu,\nu}(xy)f(y)\,dy$$

$$= \int_0^\infty f(y)\,dy\int_0^\infty x^{s-1}\varpi_{\mu,\nu}(xy)\,dx;$$

which after considerable reductions becomes

$$\tau(s) = 2^{2s-1}\frac{\Gamma(\frac{1}{2}u+\frac{1}{2}s+\frac{1}{4})\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\mu-\frac{1}{2}s+\frac{3}{4})\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})}\,\tau(1-s).$$

Here again we can easily justify the changes in the orders of integrations.

Let

$$\tau(s) = 2^{s}\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\,\psi(s),$$

then

$$\psi(s) = \psi(1-s).$$

We thus obtain a general solution, *viz.*,

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s}\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\,\psi(s)\,x^{-s}\,ds \qquad (E_1)$$

where $\psi(s)$ satisfies the above condition and certain other conditions mentioned in the next theorem.

Theorem VI. *We shall say that $f(x)$ belong to $A(\alpha, a)$ where $0<\alpha\leq\pi$, $a<\frac{1}{2}$; if it is analytic and regular in the region defined by $r>0$, and $|\theta|<\alpha$ and if it is $O[|x|^{-a-\epsilon}]$ for small x and $O[|x|^{a-1+\epsilon}]$ for large x, for every positive ϵ and uniformly in the angle $|\theta|\leq\alpha-\eta<\alpha$.*

The necessary and sufficient condition that $f(x)$ *of* $A(\alpha, a)$ *should be its own* $R_{\mu,\nu}$ *transform is that it should be of the form* (E_1), *where* $\psi(s)$ *is regular and satisfies the relation* $\psi(s) = \psi(1-s)$ *in the strip* $a < \sigma < 1-a$ *and is of the order of* $O[e^{(\frac{1}{2}\pi-\alpha+\eta)|t|}]$ $|\theta| \leq \alpha-\eta < \alpha$.

The proof follows easily as in Titchmarsh (1937).

Theorem VII. *If* $f(x)$ *defined above is* $R_{\mu,\nu}$,

then

$$g(x) = x^{-3/2}\int_0^\infty y^{\frac{1}{2}}J_{\frac{1}{2}\mu}\left(\frac{y^2}{2x^2}\right)f(y)\,dy \quad \text{is} \quad R_\nu.$$

We have

$$g(x) = \frac{1}{2\pi i}\,x^{-3/2}\int_0^\infty y^{\frac{1}{2}}J_{\frac{1}{2}\mu}\left(\frac{y^2}{2x^2}\right)dy\int_{c-i\infty}^{c+i\infty} 2^s\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})y^{-s}\psi(s)ds$$

$$= \frac{1}{2\pi i}\,x^{-3/2}\int_{c-i\infty}^{c+i\infty} 2^s\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\psi(s)ds\int_0^\infty y^{\frac{1}{2}-s}J_{\frac{1}{2}\mu}\left(\frac{y^2}{2x^4}\right)dy$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\psi_1(s)x^{-s}\,ds$$

where

$$\psi_1(s) = \frac{2^{\frac{1}{2}\mu-5/4}}{\pi^{\frac{1}{2}}}\Gamma(\tfrac{1}{4}\mu-\tfrac{1}{4}s+\tfrac{3}{8})\Gamma(\tfrac{1}{4}\mu+\tfrac{1}{4}s+\tfrac{1}{8})\psi(s)$$

and

$$\psi_1(s) = \psi_1(1-s).$$

The converse is also true.

The various changes in the order of integration are easily justifiable. §

Example. If $f(x) = K_0(x)x^{\frac{1}{2}}$; $R_{0,0}$,

then

$$g(x) = x^{-3/2}\int_0^\infty yJ_0\left(\frac{y^2}{2x^2}\right)k_0(y)\,dy \quad \text{is} \quad R_0$$

on simplification, we get the well-known result

$$g(x) = cx^{\frac{1}{2}}[H_0(\tfrac{1}{2}x^2)-Y_0(\tfrac{1}{2}x^2)]$$

which is is R_0, where c is a constant.

Theorem VIII. *If* $f(x)$ *is* $R_{\mu,\nu}$, *then*

$$g(x) = x^{-\mu-3/2}\int_0^\infty y^{\mu+\frac{1}{2}}e^{-\frac{1}{4}(y^2/x^2)}f(y)\,dy$$

is R_ν.

Since $f(x)$ is $R_{\mu,\nu}$, we have

$$g(x) = x^{-\mu-3/2}\int_0^\infty y^{\mu+\frac{1}{2}}e^{-\frac{1}{4}(y^2/x^2)}f(y)\,dy$$

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§ Writing $s=c+it$, $c>\frac{1}{2}$, we can easily prove that under the conditions assumed, the t-integral and the y-integral are absolutely convergent and the repeated integral is also absolutely convergent.

where

$$f(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s}\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\psi(s)y^{-s}\,ds$$

Hence

$$g(x) = \frac{1}{2\pi i}\,x^{-\mu-3/2}\int_{0}^{\infty} y^{\mu+\frac{1}{2}}e^{-\frac{1}{4}(y^2/x^2)}\,dy\int_{0}^{\infty} 2^{s}(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\psi(s)y^{-s}ds$$

$$= \frac{1}{2\pi i}\,x^{-\mu-3/2}\int_{c-i\infty}^{c+i\infty} 2^{s}\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\psi(s)\,ds\,\times$$

$$\times\int_{0}^{\infty} y^{\mu+\frac{1}{2}-s}e^{-\frac{1}{2}(y^2/x^2)}\,dy$$

$$= \frac{1}{2\pi i}\,2^{\frac{1}{2}\mu+\frac{1}{4}}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\mu-\tfrac{1}{2}s+\tfrac{3}{4})\psi(s)x^{-s}\,ds$$

$$\equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\psi_1(s)x^{-s}\,ds$$

where

$$\psi_1(s) = 2^{\frac{1}{2}\mu+\frac{1}{4}}\psi(s)\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}s+\tfrac{3}{4})$$

and $\psi_1(s) = \psi_1(1-s)$ and $\psi_1(s)$ satisfies the requisite conditions as mentioned in theorem VI. The conditions for the changes in the orders of integration are easily seen to be satisfied.

Example. If

$$f(t) = J_{\mu}(t)\,t^{\frac{1}{2}};\quad R_{\mu,\mu},$$

then

$$g(x) = x^{\mu+\frac{1}{2}}e^{-\frac{1}{2}x^2} \text{ is } R_{\mu}.$$

6. We shall now show that the kernel we have chosen plays the role of a transform. Let $\mu,\ \nu > -\frac{1}{2}$.

We have

$$g(x) = \int_{0}^{\infty} \varpi_{\mu,\nu}(xy)f(y)\,dy.$$

Then

$$I = \int_{0}^{\infty} \varpi_{\mu,\nu}(xy)g(y)\,dy$$

$$= \int_{0}^{\infty}\int_{0}^{\infty} \varpi_{\mu,\nu}(xy)\varpi_{\mu,\nu}(yz)f(z)\,dz\,dy$$

$$= \int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}(xy)^{\frac{1}{2}}(yz)^{\frac{1}{2}}J_{\nu}(t)J_{\nu}(T)J_{\mu}\left(\frac{xy}{t}\right)J_{\mu}\left(\frac{yz}{T}\right)f(z)\frac{dt}{t}\,\frac{dT}{T}\,dz\,dy.$$

We take $f(z)$ to be continuous and absolutely integrable in $(0, \infty)$.

Let

$$\psi(y) = \int_{0}^{\infty}(yz)^{\frac{1}{2}}J_{\mu}(yz)f(z)\,dz.$$

Then $\psi(y)$ is continuous in $y \geq 0$. We assume that it is also absolutely integrable in $(0, \infty)$.

Also we note that $(xy)^{\frac{1}{2}}\varpi_{\mu,\nu}(xy)$ is always bounded.

1. Let us first integrate with respect to t.

$$I=\int_0^\infty\int_0^\infty\int_0^\infty (yz)^{\frac{1}{2}}\varpi_{\mu,\nu}(xy)J_\nu(T)J_\mu\left(\frac{yz}{T}\right)f(z)\,\frac{dT}{T}\,dz\cdot dy$$

$$=\int_0^\infty \varpi_{\mu,\nu}(xy)\,dy\left\{\int_0^\infty\int_0^\infty J_\nu(T)J_\mu\left(\frac{yz}{T}\right)(yz)^{\frac{1}{2}}f(z)\,\frac{dT}{T}\,dz\right\}$$

$$=\int_0^\infty \varpi_{\mu,\nu}(xy)\,dy\left\{\int_0^\infty\int_0^\infty \left(\frac{yz}{T}\right)^{\frac{1}{2}}J_\nu(T)J_\mu\left(\frac{yz}{T}\right)f(z)\,\frac{dT}{\sqrt{T}}\,dz\right\}.$$

We shall now change the order of integration with respect to T and z. We consider the double integral within the second brackets. Since $x^{\frac{1}{2}}J_\nu(x)$ is bounded when $\nu \geq -\frac{1}{2}$ and $x \geq 0$, we note

(i) that the integrand is a continuous function of z and T, when the range of integration with respect to T varies from δ to ∞.

(ii) $$\left|f(z)\int_{T_1}^{T_2} J_\nu(T)\left(\frac{yz}{T}\right)^{\frac{1}{2}}J_\mu\left(\frac{yz}{T}\right)\frac{dT}{T^{\frac{1}{2}}}\right| \leq \left|kf(z)\int_{T_1}^{T_2}\frac{J_\nu(T)\,dT}{T^{\frac{1}{2}}}\right| < \epsilon, \quad (T_2 > T_1 \geq T_0),$$

the same T_0 serving for every z in $(0, a)$ where a is arbitrary.

(iii) $$\left|\int_{Z_1}^{Z_2}\left(\frac{yz}{T}\right)^{\frac{1}{2}}J_\mu\left(\frac{yz}{T}\right)f(z)\frac{J_\nu(T)}{T^{\frac{1}{2}}}dz\right| \leq \left|k\int_{Z_1}^{Z_2} f(z)\,dz\right| < \epsilon \quad (Z_2 > Z_1 \geq Z_0),$$

the same Z_0 serving for every T in (δ, b) where b is arbitrary

(iv) $$\left|\int_{T_1}^{T_2} J_\nu(T)\,\frac{dT}{T^{\frac{1}{2}}}\int_0^z\left(\frac{yz}{T}\right)^{\frac{1}{2}}J_\mu\left(\frac{yz}{T}\right)f(z)\,dz\right| \leq \left|k\int_{T_1}^{T_2}\frac{J_\nu(T)}{T^{\frac{1}{2}}}dT\int_0^z f(z)\,dz\right|$$

$$< \epsilon; \quad (T_2 > T_1 \geq T_0),$$

the same T_0 serving for every z in an unlimited interval $Z_1 \geq 0$.

(v) We can also easily prove that.

$$\int_0^\delta \frac{J_\nu(T)\,dT}{T^{\frac{1}{2}}}\int_0^\infty\left(\frac{yz}{T}\right)^{\frac{1}{2}}J_\mu\left(\frac{yz}{T}\right)f(z)\,dz \to 0 \text{ as } \delta \to 0.$$

2. I now becomes

$$=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (xy)^{\frac{1}{2}}(yz)^{\frac{1}{2}}J_\nu(t)J_\mu\left(\frac{xy}{t}\right)J_\nu(T)J_\mu\left(\frac{yz}{T}\right)f(z)\,\frac{dt}{t}\,dz\,\frac{dT}{T}\,dy$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (xy)^{\frac{1}{2}}(yz)^{\frac{1}{2}}J_\nu(t)J_\mu\left(\frac{xy}{t}\right)J_\nu(T)J_\mu\left(\frac{yz}{T}\right)f(z)\,dz\,\frac{dt}{t}\,\frac{dT}{T}\,dy$$

since z and t are independent variables and no term contains z and t together.

Integrating first with respect to s and then with respect to t_1 which we obviously can do, we get

$$I = \int_0^\infty \int_0^\infty J_\nu(T) \varpi_{\mu,\nu}(xy) \psi(y/T) \frac{dT}{T^{\frac{1}{2}}}\, dy$$

$$= \int_0^\infty \int_0^\infty J_\nu(T) \varpi_{\mu,\nu}(xy\ T)\ T^{\frac{1}{2}} \psi(y)\, dT\, dy$$

We note

(i) the integrand is a continuous function of y and T, the ranges of integration with respect to y T being (ν, ∞) and (δ, ∞) respectively.

(ii) $$\left| J_\nu(T) T^{\frac{1}{2}} \int_{Y_1}^{Y_2} \varpi_{\mu,\nu}(xy\ T)(yT)^{\frac{1}{2}} \frac{\psi(y)}{y^{\frac{1}{2}}}\, dy \right| \leq \left| k_1 J_\nu(T) T^{\frac{1}{2}} \int_{Y_1}^{Y_1} \psi(y) y^{-\frac{1}{2}} dy \right| < \varepsilon_1$$

$$(Y_2 > Y_1 \geq Y_0),$$

the same Y_0 serving for every T in (δ, μ) where b is arbitrary.

(iii) $$\left| \psi(y) y^{-\frac{1}{2}} \int_{T_1}^{T_2} J_\nu(T) T^{\frac{1}{2}} \varpi_{\mu,\nu}(xy\ T)(yT)^{\frac{1}{2}}\, dT \right| \leq \left| k_1 \psi(y) y^{-\frac{1}{2}} \int_{T_1}^{T_2} J_\nu(T)\, T^{\frac{1}{2}}\, dT \right| < \varepsilon,$$

$$(T_2 > T_1 \geq T_0)$$

the same T_0 serving for every y in (γ, c) where c is arbitrary.

(iv) $$\left| \int_{T_1}^{T_2} J_\nu(T) T^{\frac{1}{2}}\, dT \int_\nu^y \psi(y) y^{-\frac{1}{2}} (yT)^{\frac{1}{2}} \varpi_{\mu,\nu}(xy\ T)\, dy \right| < \varepsilon; \quad (T_2 > T_1 \geq T_0)$$

the same T_0 serving for every y in the unlimited interval of y, $y \geq \nu$

We can easily prove that

(v) $$\left| \int_\delta^\infty J_\nu(T) T^{\frac{1}{2}}\, dT \int_0^\nu \psi(y) y^{-\frac{1}{2}} (yT)^{\frac{1}{2}} \varpi_{\mu,\nu}(xy\ T)\, dy \right| \qquad \longrightarrow 0 \text{ as } \nu \longrightarrow 0$$

and (vi) $$\left| \int_0^\delta J_\nu(T) T^{\frac{1}{2}}\, dT \int_0^\infty \varpi_{\mu,\nu}(xy\ T)(yT)^{\frac{1}{2}} \psi(y) y^{-\frac{1}{2}}\, dy \right| \leq \left| k_1 \int_0^\delta J_\nu(T) T^{\frac{1}{2}}\, dT \int_0^\infty \psi(y) y^{-\frac{1}{2}}\, dy \right|$$

$$= \left| \int_0^\delta J_\nu(T) T^{\frac{1}{2}}\, dT\ k_1 \left\{ \int_0^1 \psi(y) y^{-\frac{1}{2}}\, dy + \int_1^\infty \psi(y) y^{-\frac{1}{2}}\, dy \right\} \right| \qquad \longrightarrow 0 \text{ as } \delta \longrightarrow 0.$$

3. I now becomes equal to

$$= \int_0^\infty \int_0^\infty \int_0^\infty \left\{ (xy)^{\frac{1}{2}} J_\nu(T) J_\nu(t) J_\mu\left(\frac{xy}{t}\right) \psi(y/T) \frac{dt}{t}\, dy \right\} \frac{dT}{T^{\frac{1}{2}}}$$

We shall now consider the changes in the order of integration between y and t. The range of integration with reepect to t is δ to ∞. The integrand within the second brackets is

$$\int_\delta^\infty \int_0^\infty \left(\frac{xy}{t}\right)^{\frac{1}{2}} J_\mu(t) t^{-\frac{1}{2}} \psi(y/T)\, dt\, dy.$$

We note

(i) the integrand is continuous in (t, y)

(ii) $$\left|\psi(y/T)\int_{t_1}^{t_2}\left(\frac{xy}{t}\right)^{\frac{1}{2}}J_\mu\left(\frac{xy}{t}\right)J_\nu(t)\,t^{-\frac{1}{2}}\,dt\right| \leq \left|k_2\int_{t_1}^{t_2}J_\nu(t)\,t^{-\frac{1}{2}}\,dt\right| < \varepsilon;\quad (t_2 > t_1 \geq t_0),$$

the same t_0 serving for every y in $(0, b)$ where b is arbitrary.

(iii) $$\left|J_\nu(t)t^{-\frac{1}{2}}\int_{Y_1}^{Y_2}\left(\frac{xy}{t}\right)^{\frac{1}{2}}J_\mu\left(\frac{xy}{t}\right)\psi(y/T)\,dy\right| \leq \left|k_2\,J_\nu(t)\,t^{-\frac{1}{2}}\int_{Y_1}^{Y_2}\psi(y/T)\,dy\right| < \varepsilon;$$
$$(Y_2 > Y_1 \geq Y_0),$$

the same Y_0 serving for every t in $(0, d)$, where d is arbitrary.

(iv) $$\left|\int_{t_1}^{t_2}J_\nu(t)\,t^{-\frac{1}{2}}\,dt\int_0^{Y}\left(\frac{xy}{t}\right)^{\frac{1}{2}}J_\mu\left(\frac{xy}{t}\right)\psi(y/T)\,dy\right| \leq \left|k\int_{t_1}^{t_2}J_\nu(t)\,t^{-\frac{1}{2}}\,dt\int_0^{Y}\psi(y/T)\,dy\right|$$
$$< \varepsilon;\quad (t_2 > t_1 \geq t_0),$$

the same t_0 serving for every y, $y \geq 0$.

On changing the order of integration, we get

$$\int_\delta^\infty J_\nu(t)\,t^{-\frac{1}{2}}\,dt\int_0^\infty\left(\frac{xy}{t}\right)^{\frac{1}{2}}J_\mu\left(\frac{xy}{t}\right)\psi(y/T)\,dy.$$

We can easily prove that

$$\int_0^\delta J_\nu(t)\,t^{-\frac{1}{2}}\,dt\int_0^\infty\left(\frac{xy}{t}\right)^{\frac{1}{2}}J_\mu\left(\frac{xy}{t}\right)\psi(y/T)\,dy \qquad \longrightarrow 0 \text{ as } \delta \longrightarrow 0.$$

Hence, ultimately

$$I = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (xy)^{\frac{1}{2}}(yz)^{\frac{1}{2}}\,J_\nu(t)J_\nu(T)J_\mu\left(\frac{xy}{t}\right)J_\mu\left(\frac{yz}{T}\right)f(z)\,dz\,dy\,\frac{dt}{t}\,\frac{dT}{T}\,.$$

Following Watson (1931), we now replace y by yT and we have

$$\int_0^\infty\int_0^\infty\left(\frac{xy}{t}\right)^{\frac{1}{2}}J_\mu\left(\frac{xy}{t}\right)\left(\frac{yz}{T}\right)^{\frac{1}{2}}J_\mu\left(\frac{yz}{T}\right)f(z)\,dz\,dy$$
$$= T\int_0^\infty\int_0^\infty\left(\frac{xy\,T}{t}\right)^{\frac{1}{2}}J_\mu\left(\frac{xy\,T}{t}\right)(yz)^{\frac{1}{2}}J_\mu(yz)\,f(z)\,dz\,dy$$
$$= Tf(xT/t),$$

by Hankel's theorem.

Hence $$\int_0^\infty \varpi_{\mu,\nu}(xy)g(y)\,dy = \int_0^\infty\int_0^\infty (t\,T)^{\frac{1}{2}}J_\nu(t)J_\nu(T)f(xT/t)\,dt\,dT/t$$
$$= \int_0^\infty\int_0^\infty (ux\,T)^{\frac{1}{2}}J_\nu(ux)J_\nu(T)f(T/u)\,dT\,du/u$$
$$= \int_0^\infty\int_0^\infty (ux)^{\frac{1}{2}}(uv)^{\frac{1}{2}}J_\nu(ux)J_\nu(uv)f(v)\,dv\,du$$
$$= f(x),$$

again by Hankel's theorem.

Consequently we prove here completely that $f(x)$ and $g(x)$ are reciprocal functions in the transform with the kernel $\varpi_{\mu,\nu}(xy)$, when $\mu,\ \nu > -\frac{1}{2}$. When $g(x) = f(x)$, the function becomes self-reciprocal.

My best thanks are due to Dr. S. C. Mitra for his suggestions and guidance in the preparation of this paper.

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CERTAIN THEOREMS ON SELF-RECIPROCAL RELATIONSHIP IN OPERATIONAL CALCULUS—II

By

B. N. Bose, *Calcutta*.

(*Received—May 18, 1958*).

If $\varphi(p)$ be the Laplace transform of a function $f(t)$, then

$$\varphi(p) = p\int_0^\infty e^{-pt} f(t)\,dt, \tag{i}$$

where p is a positive number (or a number whose real part is positive). We usually say that $\varphi(p)$ is operationally related to $f(t)$ and $f(t)$ is the interpretation of $\varphi(p)$. Following McLachlan and Humbert, we write symbolically

$$\varphi(p) \subset f(t), \text{ or } f(t) \supset \varphi(p).$$

We have recently established certain results (Bose, 1952) bearing on self-reciprocal relationship in the sine and cosine transform between functions which are connected by a set of successive operational relations. In the present paper we shall firstly obtain an extension of these results. Secondly, we shall generalise some of the theorems on self-reciprocal functions in the Hankel transform, whice has been established elsewhere in a paper (Bose, 1954).*

Following Hardy and Titchmarsh we shall say that if a function be self-reciprocal in the Hankel transform of order ν, then it is R_ν and further that a function is R_c or R_s according as it is $R_{-\frac{1}{2}}$ or $R_{\frac{1}{2}}$.

The results of the Operational Calculus required in the sequel are the following:

$$p\varphi(p) \subset \frac{d}{dt} f(t), \text{ if } f(0) = 0. \tag{ii}$$

$$\frac{\varphi(p)}{p} \subset \int_0^t f(t)\,dt \tag{iii}$$

$$(-1)^n \left(p\frac{d}{dp}\right)^n \varphi(p) \subset \left(t\frac{d}{dt}\right)^n f(t) \tag{iv}$$

Here, as in the papers referred to above, we shall make use of the well-kown Goldstein's theorem (Goldstein, 1932) and Lerch's theorem (Lerch, 1903).

Let us suppose

$$\varphi(p) \subset f_0(t),\ f_0(p) \subset f_1(t), \ldots\ldots,\ f_{n-1}(p) \subset f_n(t)$$

---

*The theorems obtained can be proved by Mellin Transforms but their converses do not seem to have been proved anywhere. Further the corollaries do not seem to have attracted any attention as yet.

We have proved before (Bose, 1952) that

$$t\int_0^\infty f_0(x)\sin xt\,dx \supset p\int_0^\infty \varphi(x)\sin px\,dx, \tag{1}$$

provided the convergence of integrals referred to there holds good.

Then from the relation

$$f_{2r}(p)\subset f_{2r+1}(t),\quad r=0,1,2,3,\ldots$$

we have, by virtue of (*iv*),

$$(-1)^r\left(p\frac{d}{dp}\right)^r f_{2r}(p)\subset\left(t\frac{d}{dt}\right)^r f_{2r+1}(t),\quad r=0,1,2,\ldots \tag{v}$$

Hence proceeding as in (1), we get

$$t\int_0^\infty\left(x\frac{d}{dx}\right)^r f_{2r+1}(x)\sin xt\,dx\supset(-1)^r p\int_0^\infty\left(x\frac{d}{dx}\right)^r f_{2r}(x)\sin px\,dx, \tag{2}$$

provided the convergence of integrals similar to those referred to in (1) holds good.

Now, let $\dfrac{d}{dx}\left(x\dfrac{d}{dx}\right)^r f_{2r}(x)$ be R_c; then we shall prove that $\left(x\dfrac{d}{dx}\right)^{r+1} f_{2r+1}(x)$ is R_s, where $r=0,1,2,\ldots.$ (A)

On integrating the right hand side of (2) by parts, we get

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} t\int_0^\infty\left(x\frac{d}{dx}\right)^r f_{2r+1}(x)\sin xt\,dx$$

$$\supset(-1)\left\{\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[\left(x\frac{d}{dx}\right)^r f_{2r}(x)\cos px\right]_{x=0}^{\infty}+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty\frac{d}{dx}\left(x\frac{d}{dx}\right)^r f_{2r}(x)\cos px\,dx\right\}$$

$$=(-1)^r\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty\frac{d}{dx}\left(x\frac{d}{dx}\right)^r f_{2r}(x)\cos px\,dx,\ \text{provided}\ \underset{\substack{x\to 0\\ \&\, x\to\infty}}{Lt}\left[\left(x\frac{d}{dx}\right)^r f_{2r}(x)\right]=0.$$

By hypothesis, we therefore get

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} t\int_0^\infty\left(x\frac{d}{dx}\right)^r f_{2r+1}(x)\sin xt\,dx\supset(-1)^r\frac{d}{dx}\left(p\frac{d}{dp}\right)^r f_{2r}(p). \tag{3}$$

Again integrating the left hand side of (3) by parts, we get

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty\frac{d}{dx}\left(x\frac{d}{dx}\right)^r f_{2r+1}(x)\cos xt\supset(-1)^r\frac{d}{dp}\left(p\frac{d}{dp}\right)^r f_{2r}(p), \tag{4}$$

provided $$\underset{x\to 0}{Lt}\left[\left(x\frac{d}{dx}\right)^r f_{2r+1}(x)\right]=\underset{x\to\infty}{Lt}\left[\left(x\frac{d}{dx}\right)^r f_{2r+1}(x)\right]=0.$$

Writing the relation (v) as

$$(-1)^r p\frac{d}{dp}\left(p\frac{d}{dp}\right)^r f_{2r}(p)\subset -\left(t\frac{d}{dt}\right)^{r+1} f_{2r+1}(t)$$

and then applying formula (iii), we get

$$(-1)^r \frac{d}{dp}\left(p\frac{d}{dp}\right)^r f_{2r}(p)\subset -\int_0^t \left(t\frac{d}{dt}\right)^{r+1} f_{2r+1}(t)\,dt. \qquad (vi)$$

Therefore by Lerch's theorem, provided $\left(x\frac{d}{dx}\right)^{r+1} f_{2r+1}(x)$ is continuous in the arbitrary interval $(0, X)$, we get from (3) and (vi)

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \frac{d}{dx}\left(x\frac{d}{dx}\right)^r f_{2r+1}(x)\cos xt\,dx = -\int_0^t \left(t\frac{d}{dt}\right)^{r+1} f_{2r+1}(t)\,dt.$$

Hence differentiating both sides with respect to t; we get, provided the differentiation on the left hand side is permissible,

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty x\frac{d}{dx}\left(x\frac{d}{dx}\right)^r f_{2r+1}(x)\sin xt\,dx = \left(t\frac{d}{dt}\right)^{r+1} f_{2r+1}(t)$$

i.e.,

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \left(x\frac{d}{dx}\right)^{r+1} f_{2r+1}(x)\sin xt\,dx = \left(t\frac{d}{dx}\right)^{r+1} f_{2r+1}(t), \qquad (5)$$

which shows that $\left(x\frac{d}{dx}\right)^{r+1} f_{2r+1}(x)$ is R_s.

We have thus proved what has been stated in (A)

Assuming that $\left(x\frac{d}{dx}\right)^{r+1} f_{2r+1}(x)$ is R_s, we next proceed to show that

$$\frac{d}{dx}\left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x) \text{ is } R_c, \text{ where } r = 0, 1, 2, \ldots\ldots \qquad (B)$$

We have, as in (v),

$$\left(p\frac{d}{dp}\right)^{r+1} f_{2r+1}(p)\subset(-1)^{r+1}\left(t\frac{d}{dt}\right)^{r+1} f_{2r+2}(t),\ r = 0, 1, 2, \ldots\ldots \qquad (vii)$$

and then proceeding as in (2), we get

$$t\int_0^\infty \left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x)\sin xt\,dx\supset(-1)^{r+1}p\int_0^\infty \left(x\frac{d}{dx}\right)^{r+1} f_{2r+1}(x)\sin px\,dx,\ r = 0, 1, 2, \ldots \qquad (6)$$

provided the convergence of certain integrals similar to those referred to in (2) holds good.

Therefore, by hypothesis, we have

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x)\sin xt\,dx\supset(-1)^{r+1}p\left(p\frac{d}{dp}\right)^{r+1} f_{2r+1}(p).$$

Hence integrating the left hand side by parts, we get

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \frac{d}{dx}\left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x)\cos xt\,dx \supset (-1)^{r+1}\,p\left(p\frac{d}{dp}\right)^{r+1} f_{2r+1}(p), \tag{7}$$

provided $\underset{x\to 0}{Lt}\left[\left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x)\right] = \underset{x\to\infty}{Lt}\left[\left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x)\right] = 0.$

But from (*vii*), we have, by virtue of (*ii*)

$$(-1)^{r+1}\,p\left(p\frac{d}{dp}\right)^{r+1} f_{2r+1}(p) \subset \frac{d}{dt}\left[\left(t\frac{d}{dt}\right)^{r+1} f_{2r+2}(t)\right], \tag{viii}$$

provided $\underset{x\to 0}{Lt}\left[\left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x)\right] = 0.$

Therefore, provided $\frac{d}{dx}\left[\left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x)\right]$ is continuous in $(0, X)$, by Lerch's theorem, we get, from (7) and (viii)

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \frac{d}{dx}\left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x)\cos xt\,dx = \frac{d}{dt}\left(t\frac{d}{dt}\right)^{r+1} f_{2r+2}(t), \tag{8}$$

which shows that $\frac{d}{dx}\left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}$ is R_c.

What we have stated in (B) is thus proved.

Let us suppose that $\varphi(x)$ is R_s. Then we have proved before (Bose, 1952) that

$$\frac{d}{dx}f_0(x) \text{ is } R_c.$$

Therefore, by virtue of (A), it follows that

$$x\frac{d}{dx}f_1(x) \text{ is } R_s; \tag{9}$$

whence, by (B),

$$\frac{d}{dx}\left(x\frac{d}{dx}\right)f_2(x) \text{ is } R_c \tag{10}$$

and therefore, again by (A),

$$\left(x\frac{d}{dx}\right)^2 f_3(x) \text{ is } R_s \tag{11}$$

and so on. Thus we have

Theorem I: *Let* $\varphi(p)\subset f_0(t)$ *and* $f_r(p)\subset f_{r+1}(t)$, $r = 0, 1, 2, \ldots n-1$ *and* $\underset{x\to 0 \text{ and } x\to\infty}{Lt}\left[\left(x\frac{d}{dx}\right)^r f_{2r}(x)\right] = \underset{x\to 0 \text{ and } x\to\infty}{Lt}\left[\left(x\frac{d}{dx}\right)^r f_{2r+1}(x)\right] = 0.$ *Further let* $\left(x\frac{d}{dx}\right)^{r+1} f_{2r+1}(x)$ *and* $\frac{d}{dx}\left[\left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x)\right]$ *be continuous in* $(0, X)$. *Then if* $\varphi(x)$ *be* R_s, $\frac{d}{dx}\left(x\frac{d}{dx}\right)^{\frac{1}{2}n} f_n(x)$ *is* R_c *when* n *is even and* $\left(x\frac{d}{dx}\right)^{\frac{1}{2}(n+1)} f_n(x)$ *is* R_s *when* n *is odd.*

Again, we have proved before (Mittra and Bose, 1953) that

$$t\int_0^\infty f_0(x)\cos xt\,dx \supset -p\int_0^\infty \varphi(x)\cos px\,dx \tag{12}$$

Then we first deduce from (12), the relations

$$t\int_0^\infty \left(x\frac{d}{dx}\right)^r f_{2r+1}(x)\cos xt\,dx \supset (-1)^{r+1}\,p\int_0^\infty \left(x\frac{d}{dx}\right)^r f_{2r}(x)\cos px\,dx,\quad r=0,\,1,\,2,\ldots \tag{13}$$

and $$t\int_0^\infty \left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x)\cos xt\,dx \supset (-1)^{r+1}\,p\int_0^\infty \left(x\frac{d}{dx}\right)^{r+1} f_{2r+1}(x)\cos px\,dx,$$

$$r=0,\,1,\,2,\ldots \tag{14}$$

and then proceeding as before, we establish

Theorem II : *Let* $\varphi(p)\subset f_0(t)$ *and* $f_r(p)\subset f_{r+1}(t)$, $r=0,\,1,\ldots,\,n-1$ *and* $\underset{x\to 0 \text{ and } x\to\infty}{Lt}\left[\left(x\frac{d}{dx}\right)^r f_{2r}(x)\right]=\underset{x\to 0 \text{ and } x\to\infty}{Lt}\left[\left(x\frac{d}{dx}\right)^r f_{2r+1}(x)\right]=0.$ *Further let* $\left(x\frac{d}{dx}\right)^{r+1} f_{2r+1}(x)$ *and* $\frac{d}{dx}\left[\left(x\frac{d}{dx}\right)^{r+1} f_{2r+2}(x)\right]$ *be continuous in* $(0,\,X)$. *Then if* $\varphi(x)$ *be* R_c, $\frac{d}{dx}\left(x\frac{d}{dx}\right)^{\frac{1}{2}n} f_n(x)$ *is* R_c *when* n *is even and* $\left(x\frac{d}{dx}\right)^{\frac{1}{2}(n+1)} f_n(x)$ *is* R_c *when* n *is odd.*

Conversely, let us first suppose that n is even, say $2m$, and $\frac{d}{dx}\left(x\frac{d}{dx}\right)^m f_{2m}(x)$ is R_c. Then we have

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \frac{d}{dx}\left(x\frac{d}{dx}\right)^m f_{2m}(x)\cos xt\,dx=\frac{d}{dt}\left(t\frac{d}{dt}\right)^m f_{2m}(t) \tag{15}$$

But since

$$\left(t\frac{d}{dt}\right)^m f_{2m}(t)\supset(-1)^m\left(p\frac{d}{dp}\right)^m f_{2m-1}(p),$$

by applying (ii), we get

$$\frac{d}{dt}\left(t\frac{d}{dt}\right)^m f_{2m}(t)\supset(-1)^m p\left(p\frac{d}{dp}\right)^m f_{2m-1}(p),$$

provided

$$\underset{x\to 0}{Lt}\left[\left(x\frac{d}{dx}\right)^m f_{2m}(x)\right]=0.$$

From (15), we therefore get

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \frac{d}{dx}\left(x\frac{d}{dx}\right)^m f_{2m}(x)\cos xt\,dx\supset(-1)^m p\left(p\frac{d}{dp}\right)^m f_{2m-1}(p) \tag{16}$$

But $$\int_0^\infty \frac{d}{dx}\left(x\frac{d}{dx}\right)^m f_{2m}(x)\cos xt\,dx=t\int_0^\infty\left(x\frac{d}{dx}\right)^m f_{2m}(x)\sin xt\,dx,$$

provided $$\underset{x\to 0}{Lt}\left[\left(x\frac{d}{dx}\right)^m f_{2m}(x)\right]=\underset{x\to\infty}{Lt}\left[\left(x\frac{d}{dx}\right)^m f_{2m}(x)\right]=0.$$

Therefore $$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} t\int_0^\infty \left(x\frac{d}{dx}\right)^m f_{2m}(x)\sin xt\,dx \supset (-1)^m p\left(p\frac{d}{dp}\right)^m f_{2m-1}(p) \tag{17}$$

But by (6), we have

$$t\int_0^\infty \left(x\frac{d}{dx}\right)^m f_{2m}(x)\sin xt\,dx \supset (-1)^m p\int_0^\infty \left(x\frac{d}{dx}\right)^m f_{2m-1}(x)\sin px\,dx.$$

Hence from (17), we get

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \left(x\frac{d}{dx}\right)^m f_{2m-1}(x)\sin px\,dx=\left(p\frac{d}{dp}\right)^m f_{2m-1}(p), \tag{18}$$

which shows that $$\left(x\frac{d}{dx}\right)^m f_{2m-1}(x) \text{ is } R_s.$$

We thus prove that

if $$\frac{d}{dx}\left(x\frac{d}{dx}\right)^m f_{2m}(x) \text{ be } R_c, \text{ then } \left(x\frac{d}{dx}\right)^m f_{2m-1}(x) \text{ is } R_s. \tag{C}$$

Writing (18) as

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty x\frac{d}{dx}\left(x\frac{d}{dx}\right)^{m-1} f_{2m-1}(x)\sin xt\,dx=\left(t\frac{d}{dt}\right)^m f_{2m-1}(t)$$

and then integrating both sides with respect to t between 0 and t and changing the order of integration on the left hand side, which is assumed permissible, we get

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \frac{d}{dx}\left(x\frac{d}{dx}\right)^{m-1} f_{2m-1}(x)\,(1-\cos xt)\,dx=\int_0^t \left(t\frac{d}{dt}\right)^m f_{2m-1}(t)\,dt$$

and therefore $$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \frac{d}{dx}\left(x\frac{d}{dx}\right)^{m-1} f_{2m-1}(x)\cos xt\,dx=\int_0^t \left(t\frac{d}{dt}\right)^m f_{2m-1}(t)\,dt, \tag{19}$$

provided $$\underset{x\to 0}{Lt}\left[\left(x\frac{d}{dx}\right)^{m-1} f_{2m-1}(x)\right]=\underset{x\to\infty}{Lt}\left[\left(x\frac{d}{dx}\right)^{m-1} f_{2m-1}(x)\right]=0.$$

Further integrating the left hand side of (19) by parts, we get

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} t\int_0^\infty \left(x\frac{d}{dx}\right)^{m-1} f_{2m-1}(x)\sin xt\,dx=-\int_0^t \left(t\frac{d}{dt}\right)^m f_{2m-1}(t)\,dt. \tag{20}$$

But since $$(-1)^m\left(p\frac{d}{dp}\right)^m f_{2m-2}(p)\supset\left(t\frac{d}{dt}\right)^m f_{2m-1}(t),$$

by applying (*iii*), we get

$$(-1)^m\frac{d}{dp}\left(p\frac{d}{dp}\right)^{m-1} f_{2m-2}(p)\supset\int_0^t \left(t\frac{d}{dt}\right)^m f_{2m-1}(t)\,dt.$$

Hence from (20), we have

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} t\int_0^\infty \left(x\frac{d}{dx}\right)^{m-1} f_{2m-1}(x) \sin xt\, dx \supset (-1)^{m-1}\frac{d}{dp}\left(p\frac{d}{dp}\right)^{m-1} f_{2m-2}(p)$$

Also from (2) and the steps following it, we have

$$t\int_0^\infty \left(x\frac{d}{dx}\right)^{m-1} f_{2m-1}(x) \sin xt\, dx \supset (-1)^{m-1}\int_0^\infty \frac{d}{dx}\left(x\frac{d}{dx}\right)^{m-1} f_{2m-2}(x) \cos px\, dx.$$

We, therefore, get

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{d}{dx}\left(x\frac{d}{dx}\right)^{m-1} f_{2m-2}(x) \cos px\, dx = \frac{d}{dp}\left(p\frac{d}{dp}\right)^{m-1} f_{2m-2}(p) \tag{21}$$

which shows that $\frac{d}{dx}\left(x\frac{d}{dx}\right)^{m-1} f_{2m-2}(x)$ is R_c.

We thus prove that

$$\text{if } \left(x\frac{d}{dx}\right)^{m} f_{2m-1}(x) \text{ be } R_s, \text{ then } \frac{d}{dx}\left(x\frac{d}{dx}\right)^{m-1} f_{2m-2}(x) \text{ is } R_c. \tag{D}$$

Hence (C) and (D) together give that

$$\text{if } \frac{d}{dx}\left(x\frac{d}{dx}\right)^{m} f_{2m}(x) \text{ be } R_c, \text{ then } \frac{d}{dx}\left(x\frac{d}{dx}\right)^{m-1} f_{2m-2}(x) \text{ is } R_c. \tag{22}$$

Continuing the process, we finally arrive at the result that

$$\frac{d}{dx} f_0(x) \text{ is } R_c \tag{23}$$

and hence by Theorem II (Bose, 1951), we get that

$$\varphi(x) \text{ is } R_s, \tag{23}$$

provided $f(0) = f(\infty) = 0.$

Let us next suppose that n is odd, say $2m+1$, and $\left(x\frac{d}{dx}\right)^{m+1} f_{2m+1}(x)$ is R_s. Then applying the result (D) first and then the result (C) and thus continuing the process, we finally get that

$$\frac{d}{dx} f_0(x) \text{ is } R_c$$

and hence $\varphi(x)$ is R_s.

We thus have, as a converse to Theorem I,

Theorem III. *Let* $\varphi(p) \subset f_0(t)$, $f_r(p) \subset f_r(t)$, $r = 0, 1, 2, \ldots, n-1$ *and further let* $\frac{d}{dx}\left(x\frac{d}{dx}\right)^{\frac{1}{2}n} f_n(x)$ *be* R_c, *or* $\left(x\frac{d}{dx}\right)^{\frac{1}{2}(n+1)} f_n(x)$ *be* R_s, *according as* n *is even or odd, then* $\varphi(x)$ *is* R_s, *provided all the conditions referred to above hold good.*

Again, by assuming that $\frac{d}{dx}\left(x\frac{d}{dx}\right)^{\frac{1}{2}n}f_n(x)$ is R_s when n is even and $\left(x\frac{d}{dx}\right)^{\frac{1}{2}(n+1)}f_n(x)$ is R_c when n is odd and proceeding as before, we prove that $\varphi(x)$ is R_c. We thus have as a converse to Theorem II, the following:

Theorem IV. *Let* $\varphi(p)\subset f_0(t)$, $f_r(p)\subset f_{r+1}(t)$, $r = 0, 1, \ldots, n-1$ *and further let* $\frac{d}{dx}\left(x\frac{d}{dx}\right)^{\frac{1}{2}n}f_n(x)$ *be* R_s, *or* $\left(x\frac{d}{dx}\right)^{\frac{1}{2}(n+1)}f_n(x)$ *be* R_c, *according as n is even or odd, then* $\varphi(x)$ *is* R_c, *provided certain conditions similar to those required in Theorem III are satisfied.*

We may note here the following two elegant relations that hold good between $\varphi(x)$ and $f_1(x)$ and $\varphi(x)$ and $f_2(x)$ when the relations are respectively

$$\varphi(p)\subset f_0(t), \quad f_0(p)\subset f_1(t)$$

and

$$\varphi(p)\subset f_0(t), \quad f_0(p)\subset f_1(t), \quad f_1(p)\subset f_2(t).$$

We have

$$\varphi(p) = p\int_0^\infty e^{-pt}t\,dt\int_0^\infty e^{-xt}f_1(x)\,dx$$

$$= p\int_0^\infty \frac{f_1(x)\,dx}{(p+x)^2},$$

on changing the order of integration, which is assumed permissible.

Again

$$\varphi(p) = p\int_0^\infty \frac{t\,dt}{(p+t)^2}\int_0^\infty e^{-xt}f_2(x)\,dx$$

$$= p^{\frac{1}{2}}\int_0^\infty x^{-\frac{1}{2}}f_2(x)\ \exp\ (\tfrac{1}{2}px)W_{-3/2,\,0}(px)\,dx$$

on changing the order of integration, which is assumed permissible. We note that with these kernels, the self-reciprocal relationships between $\varphi(p)$ and $f_1(t)$ or $f_2(t)$ hold good.

We next proceed to generalise some of the theorems proved before (Bose, 1954), which relate to self-reciprocal relationship in the Hankel transform.

We first consider functions of a single variable and take a pair of functions $f(t)$ and $\psi(p)$ satisfying the integral equation

$$\psi(p) = \int_0^\infty (pt)^\mu k_m(pt)f(t)\,dt \tag{25}$$

Let us suppose that $p^\nu\psi(p)$ is R_{am}. Then we have

$$y^\nu\psi(y) = \int_0^\infty (py)^{\frac{1}{2}}J_{am}(py)p^\nu\psi(p)\,dp$$

$$= \int_0^\infty (py)^{\frac{1}{2}}J_{am}(py)p^\nu\,dp\int_0^\infty (px)^\mu k_m(px)f(x)\,dx$$

$$= y^{\frac{1}{2}}\int_0^\infty x^\mu\,f(x)\,dx\int_0^\infty p^{\mu+\nu+\frac{1}{2}}J_{am}(py)k_m(px)\,dp,$$

on changing the order of integration, which is assumed permissible.

Therefore we can write

$$y^{\nu}\psi(y) = y^{\frac{1}{2}}\int_0^\infty x^{\mu}f(x)\,dx\int_0^\infty p^{\mu+\nu+\frac{1}{2}}J_{am}(py)K_{-m}(px)\,dp.$$

Let $\mu+\nu = (a-1)m+\frac{1}{2}$; then we get

$$\psi(y) = 2^{(a-1)m}y^{am+\frac{1}{2}-\nu}\Gamma(am-m+1)\int_0^\infty \frac{x^{\mu-m}f(x)\,dx}{(x^2+y^2)^{(a-1)m+1}} \tag{26}$$

Again

$$\begin{aligned}\psi(y) &= \int_0^\infty (py)^{\mu}K_{-m}(py)f(p)\,dp\\ &= 2^{(a-1)m}\Gamma(am-m+1)y^{\mu+m}\int_0^\infty p^{\mu-(a-1)m}f(p)\,dp\int_0^\infty \frac{x^{(a-2)m+1}J_{(a-2)m}(px)\,dx}{(x^2+y^2)^{(a-1)m+1}}\\ &= 2^{(a-1)m}\Gamma(am-m+1)y^{am-\nu+\frac{1}{2}}\int_0^\infty \frac{x^{(a-2)m+\frac{1}{2}}\,dx}{(x^2+y^2)^{(a-1)m+1}}\int_0^\infty (px)^{\frac{1}{2}}J_{(a-2)m}(px)\times\\ &\qquad\times p^{\mu+(1-a)m-\frac{1}{2}}f(p)\,dp,\end{aligned} \tag{27}$$

on changing the order of integration again.

From (26) and (27) we therefore get

$$\int_0^\infty \frac{x^{\mu-m}f(x)\,dx}{(x^2+y^2)^{(a-1)m+1}} = \int_0^\infty \frac{x^{(a-2)m+\frac{1}{2}}F(x)\,dx}{(x^2+y^2)^{(a-1)m+1}}$$

i.e.,

$$\int_0^\infty \frac{x^{\mu+\nu-m}\{x^{-\nu}f(x)\}\,dx}{(x^2+y^2)^{\mu+\nu+\frac{1}{2}}} = \int_0^\infty \frac{x^{\nu+\mu-m}F(x)\,dx}{(x^2+y^2)^{\mu+\nu+\frac{1}{2}}}, \tag{28}$$

where

$$F(x) = \int_0^\infty (px)^{\frac{1}{2}}J_{(a-2)m}(px)p^{\mu+(1-a)m-\frac{1}{2}}f(p)\,dp$$

i.e.,

$$F(x) = \int_0^\infty (px)^{\frac{1}{2}}J_{(a-2)m}(px)p^{-\nu}f(p)\,dp.$$

Hence by Lemma 1 of the same paper referred to just before, provided $\mu+\nu-m \geqq 1$ and $m > 0$ and $x^{-\nu}f(x)$ is continuous and absolutely integrable in $(0, \infty)$, and $F(\infty)$ exists, we get

$$x^{-\nu}f(x) = F(x)$$

i.e.,

$$\int_0^\infty (px)^{\frac{1}{2}}J_{(a-2)m}(px),\ p^{-\nu}f(p)\,dp = x^{-\nu}f(x), \tag{29}$$

which shows that

$$x^{-\nu}f(x) \text{ is } R_{(a-2)m}$$

It is obvious from the relation $\mu+\nu = (a-1)m+\frac{1}{2}$ that $a > 2$.

We thus have

Theorem V. *If* $x^{\nu}\psi(x)$ *be* R_{am}, *then* $x^{-\nu}f(x)$ *is* $R_{(a-2)m}$, *where* $(a-1)m+\frac{1}{2} = \mu+\nu$, *provided* $x^{-\nu}f(x)$ *is continuous and absolutely integrable in* $(0, \infty)$ *and* $\mu+\nu-1 \geqq m > 0$.

Conversely, let $x^{-\nu}f(x)$ *be* $R_{(a-2)m}$, *where* $(a-1)m+\frac{1}{2} = \mu+\nu$. *Then we have*

$$x^{-\nu}f(x) = \int_0^\infty (px)^{\frac{1}{2}}J_{(a-2)m}(px)\, p^{-\nu}f(p)\, dp. \tag{30}$$

Proceeding as before and changing the order of integration, we get, as in (26) and (27),

$$\int_0^\infty (py)^{\frac{1}{2}}J_{am}(py)p^{\nu}\psi(p)\, dp = 2^{(a-1)m}\Gamma(am-m+1)\, y^{am+\frac{1}{2}}\int_0^\infty \frac{x^{\mu-m}f(x)\, dx}{(x^2+y^2)^{(a-1)m+1}}$$

and

$$\psi(y) = 2^{(a-1)m}\Gamma(am-m+1)y^{am+\frac{1}{2}-\nu}\int_0^\infty \frac{x^{\mu+\nu-m}\, dx}{(x^2+y^2)^{(a-1)m+1}}\int_0^\infty (px)^{\frac{1}{2}}J_{(a-2)m}(px)p^{-\nu}f(p)\, dp$$

$$= 2^{(a-1)m}\Gamma(am-m+1)y^{am+\frac{1}{2}-\nu}\int_0^\infty \frac{x^{\mu-m}f(x)\, dx}{(x^2+y^2)^{(a-1)m+1}}.$$

Hence we obtain

$$\int_0^\infty (py)^{\frac{1}{2}}J_{am}(py)p^{\nu}\psi(p)\, dp = y^{\nu}\psi(y), \tag{31}$$

thus showing that $x^{\nu}\psi(x)$ is R_{am}. We therefore have

Theorem VI. *If* $x^{-\nu}f(x)$ *be* $R_{(a-2)m}$, *then* $x^{\nu}\psi(x)$ *is* R_{am}, *where* $(a-1)m+\frac{1}{2} = \mu+\nu$, *provided the integral for the self-reciprocal property exists and* $m > 0$.

From Theorems V and VI we get as particular cases the following results established in the paper referred to:

(*a*) Theorems C and D, when $\mu = \frac{1}{2}$, $\nu = 2m$ and $a = 3$,

(*b*) Theorems E and F, when $\mu = m$, $\nu = m+\frac{1}{2}$ and $a = 3$,

(*c*) Theorems G and H, when $\mu = m+1$, $\nu = m-\frac{1}{2}$ and $a = 3$.

The following results are are easily deduced from Theorems V and VI.

We can write

$$\psi(p) = \int_0^\infty (pt)^{m+1}K_m(pt)\, (pt)^{\mu-m-1}f(t)\, dt$$

$$= -\int_0^\infty \frac{d}{d(pt)}\{(pt)^{m+1}K_{m+1}(pt)\}\, (pt)^{\mu-m-1}f(t)\, dt$$

$$= -\frac{1}{p}\int_0^\infty \frac{d}{dt}\{(pt)^{m+1}K_{m+1}(pt)\}(pt)^{\mu-m-1}f(t)\, dt,$$

Hence integrating by parts and assuming that $\underset{t\to 0}{Lt}[t^{\mu-m-1}f(t)] = 0$ and $\underset{t\to\infty}{Lt}[t^{\mu-m-1}f(t)]$ is finite $(m+1>0)$, we get

$$p\psi(p) = \int_0^\infty (pt)^{m+1}K_{m+1}(pt)\frac{d}{dt}\{(pt)^{\mu-m-1}f(t)\}\,dt$$

$$= \int_0^\infty (pt)^{\mu}K_{m+1}(pt)\;t^{m+1-\mu}\frac{d}{dt}\{t^{\mu-m-1}f(t)\}\,dt. \tag{32}$$

Applying Theorems V and VI, we get

Corollary A. If $x^{\nu+1}\psi(x)$ *be* $R_{a(m+1)}$ *then* $x^{m-\mu-\nu+1}\dfrac{d}{dx}\{x^{\mu-m-1}f(x)\}$ *is* $R_{(a-2)(m+1)}$ *and conversely, where* $(a-1)(m+1)+\frac{1}{2}=\mu+\nu$ *and provided* $\underset{t\to 0}{Lt}[t^{\mu-m}f(t)] = 0$ *and* $\underset{t\to\infty}{Lt}[t^{\mu-m}f(t)]$ *is finite and* $x^{-\nu}[x^{m+1-\mu}(d/dx)\{x^{\mu-m-1}f(x)\}]$ *is continuous and absolutely integrable in* $(0,\infty)$ *axd* $\mu+\nu-1 \geqq m+1>0$.

Again assuming that differentiation under the sign of integration is permissible, we get

$$\frac{d}{dp}[p\psi(p)] = \int_0^\infty \frac{d}{dp}\{(pt)^{\mu}K_{m+1}(pt)\}\,t^{m+1-\mu}\frac{d}{dt}\{t^{\mu-m-1}f(t)\}\,dt$$

$$= \frac{1}{p}\int_0^\infty \frac{d}{dt}\{(pt)^{\mu}K_{m+1}(pt)\}\,tF_1(t)\,dt,$$

where

$$F_1(t) = t^{m+1-\mu}\frac{d}{dt}\{t^{\mu-m-1}f(t)\}.$$

Hence, as before, we get

$$p\frac{d}{dp}\{p\psi(p)\} = -\int_0^\infty (pt)^{\mu}K_{m+1}(pt)\frac{d}{dt}\{t\,F_1(t)\}\,dt, \tag{33}$$

provided $\underset{t\to 0}{Lt}[t^{\mu-m}F_1(t)] = 0$ and $\underset{t\to\infty}{Lt}[t^{\mu-m}F_1(t)]$ is finite.

Applying theorems V and VI again, we get

Corollary B. If $x^{\nu+1}\dfrac{d}{dx}\{x\psi(x)\}$ *be* $R_{a(m+1)}$, *then* $x^{-\nu}\dfrac{d}{dx}[-x\,F_1(x)]$ *is* $R_{(a-2)(m+1)}$ *and conversely, where* $(a-1)\,(m+1)+\frac{1}{2} = \mu+\nu$ *and* $F_1(x) = x^{1+m-\mu}\dfrac{d}{dx}\{x^{\mu-m-1}f(x)\}$.

Proceeding exactly as before and assuming that

$\underset{t\to 0}{Lt}[t^{\mu-m}\dfrac{d}{dt}\{t\,F_1(t)\}] = 0$, $\underset{t\to\infty}{Lt}[t^{\mu-m}\dfrac{d}{dt}\{t\,F_1(t)\}]$ is finite, we get

Corollary C : *If* $x^{\nu+1}\frac{d}{dx}\left[x\frac{d}{dx}\{x\psi(x)\}\right]$ *be* $R_{a(m+1)}$, *then* $x^{-\nu}\frac{d}{dx}\left[x\frac{d}{dx}\{xF_1(x)\}\right]$ *is* $R_{(a-2)(m+1)}$ *and conversely.*

Thus these results can be generalised when the number of operations performed is n, under suitable conditions.

Examples of Theorems V and VI: (*i*) We have the result

$$\int_0^\infty t^{c-1}K_m(2(pt)^{\frac12})\,{}_2F_2\left[\begin{matrix}\frac12-k-m, & \frac12-k+m; & -t\\ c-\frac12 m', & c+\frac12 m' & \end{matrix}\right]dt = \frac{\Gamma(c-\frac12 m')\Gamma(c+\frac12 m')}{2}$$
$$e^{\frac12 p}p^{-(c+k)}W_{k,m}(p),\quad (c\pm\frac12 m' > 0).$$

This relation can be put in the form

$$\int_0^\infty (pt)^\mu K_{m'}(pt)t^{2c-\mu-1}{}_1F_1(c+2m+\tfrac12 m'; n-\tfrac12 m'-\tfrac12 t^2)\,dt = Ae^{\frac14 p^2}p^{\mu+2m+m'-1}$$
$$W_{\frac12-\frac12 m'-m-c,\, m}(\tfrac12 p^2),$$

where A is some constant.

On putting $(a-1)m'+\frac12 = \mu+\nu$ and $c = (a-\frac12)m'+2m+1$, we get

$$p^\nu\psi(p) = e^{\frac14 p^2}p^{am'+2m-\frac12}W_{-\frac12-am'-3m,\, m}(\tfrac12 p^2),$$
$$t^{-\nu}f(t) = t^{(a-2)m'+2\alpha+\frac12}e^{-\frac12 t^2}{}_1F_1[-\alpha; (a-2)m'+\alpha+1; \tfrac12 t^2],$$

where $\alpha = 2m+m'$.

It has been shown by Bailey (1930) that $p^\nu\psi(p)$ is $R_{am'}$, and that $t^{-\nu}f(t)$ is $R_{(a-2)m'}$ has proved by B. Mohan (1939), provided $(a-2)m'+\alpha+ +1 > 0$. This is what we should have according to the theorems. It can be easily seen that the conditions of the theorems are easily satisfied.

(*ii*) Again, we have the result (Bose, N. N., 1952)

$$\int_0^\infty t^{2\nu+1}{}_2F_2(\alpha, \beta; 1, 2\nu+1; -t^2)K_{2\nu}(2pt)\,dt = \frac{\Gamma(2\nu+1)}{4\Gamma(\alpha)\Gamma(\beta)}\frac{E(\alpha, \beta :: p^2)}{p^{2\nu+2}}.$$

This result can be easily put in the form

$$\int_0^\infty (pt)^\mu K_m(pt)t^{m+1-\mu}{}_1F_1(\alpha; 1; -\tfrac12 t^2)\,dt = A\,\frac{E(\alpha, m+1 :: \frac12 p^2)}{p^{m+2-\mu}},$$

where A' is some constant.

On putting $\mu+\nu = (a-1)m+\frac12$ and $\alpha = (2-a)m+1$, we get

$$p^\nu\psi(p) = p^{-\alpha-\frac12}E(\alpha, m+1 :: \tfrac12 p^2)$$

and

$$t^{-\nu}f(t) = t^{\alpha-\frac12}{}_1F_1(\alpha; 1: -\tfrac12 t^2).$$

It has been shown by Bailey (1931) that $t^{-\nu}f(t)$ is $R_{1-\alpha}$ and that $p^\nu\psi(p)$ is $R_{2m+1-\alpha}$ has been obtained by N. N. Bose (1950). These results are quite in accord with theorems. The conditions for the validity of the theorems are easily found to be satisfied.

We now proceed to extend Theorems V and VI to cases when the number of variables is two or more. We consider below the cases of two variables only. The nature of the proofs given will indicate that they can be easily extended to the cases of more than two variables.

Theorem VII. *Let*

$$\psi(p_1, p_2) = \int_0^\infty \int_0^\infty (p_1 t_1)^{\mu_1} (p_2 t_2)^{\mu_2} k_m(p_1 t_1) k_n(p_3 t_2) f(t_1, t_2)\, dt_1\, dt_2 \tag{34}$$

and further let $p_1^{\nu_1} p_2^{\nu_2} \psi(p_1, p_2)$ *be* R_{am} *and* $R_{a'n}$ *with respect* p_1 *and* p_2 *respectively. Then* $x^{-\nu_1} x_2^{-\nu_2} f(x_1, x_2)$ *is* $R_{(a-2)m}$ *and* $R_{(a'-2)n}$ *with respect to* x_1 *and* x_2 *respectively, where* $\mu_1 + \nu_1 = (a-1)m + \frac{1}{2}$ *and* $\mu_2 + \nu_2 = (a'-1)n + \frac{1}{2}$, *provided* $x_1^{\mu_1 - m} x_2^{\mu_2 - n} f(x_1, x_2)$ *and* $x_1^{\nu_1 + \mu_1 - m} x_2^{\nu_2 + \mu_2 - n} F(x_1, x_2)$ *are continuous in both variables in* $0 \leqq x_i \leqq \infty$, $i = 1, 2$ *and are absolutely integrable in* $(0, \infty)$ *with respect to* x_1 *and* x_2 *and further* $\int_0^\infty x_1^{\mu_1 - m} x_2^{\mu_2 - n} \mid f(x_1, x_2) \mid dx_i$ $(i = 1, 2)$ *and* $\int_0^\infty x_1^{\nu_1 + \mu_1 - m} x_2^{\nu_2 + \mu_2 - n} \mid F(x_1, x_2) \mid dx_i$, $(i = 1, 2)$ *are integrable in* $(0, \infty)$, *where* $F(x_1, x_2)$ *is defined below in (38).*

In order to prove this, we first note that by hypothesis,

$$\begin{aligned}
y_1^{\nu_1} y_2^{\nu_2} \psi(y_1, y_2) &= \int_0^\infty \int_0^\infty (p_1 y_1)^{\frac{1}{2}} (p_2 y_2)^{\frac{1}{2}} J_{am}(p_1 y_1) J_{a'n}(p_2 y_2) p_1^{\nu_1} p_2^{\nu_2} \psi(p_1, p_2)\, dp_1\, dp_2 \\
&= \int_0^\infty \int_0^\infty (p_1 y_1)^{\frac{1}{2}} (p_2 y_2)^{\frac{1}{2}} J_{am}(p_1 y_1) J_{a'n}(p_2 y_2) p_1^{\nu_1} p_2^{\nu_1}\, dp_1\, dp_2 \times \\
&\qquad \int_0^\infty \int_0^\infty (p_1 x_1)^{\mu_1} (p_2 x_2)^{\mu_2} K_m(p_1 x_1) K_n(p_2 x_2) f(x_1, x_2)\, dx_1\, dx_2 \\
&= \int_0^\infty \int_0^\infty x_1^{\mu_1} x_2^{\mu_2} f(x_1, x_2)\, dx_1\, dx_2 \times \\
&\qquad (y_1 y_2)^{\frac{1}{2}} \int_0^\infty \int_0^\infty p_1^{\mu_1 + \nu_1 + \frac{1}{2}} p_2^{\mu_2 + \nu_2 + \frac{1}{2}} J_{am}(p_1 y_1) J_{a'n}(p_2 y_2) \times \\
&\qquad\qquad K_{-m}(p_1 x_1) K_{-n}(p_2 x_2)\, dp_1\, dp_2,
\end{aligned}$$

on changing the order of integration, which is assumed permissible.

Hence we get

$$\psi(y_1, y_2) = \Gamma(am - m + 1) \Gamma(a'n - n + 1) 2^{(a-1)m(a'-1)n} y_1^{am + \frac{1}{2} - \nu_1} y_2^{a'n + \frac{1}{2} - \nu_2} \times$$

$$\int_0^\infty \int_0^\infty \frac{x_1^{\mu_1 - m} x_2^{\mu_2 - n} f(x_1, x_2)\, dx_1\, dx_2}{(x_1^2 + y_1^2)^{(a-1)(m+1)} (x_2^2 + y_2^2)^{(a'-1)n+1}}\,. \tag{35}$$

Again,

$$\psi(y_1, y_2) = \int_0^\infty \int_0^\infty (p_1y_1)^{\mu_1}(p_2y_2)^{\mu_2}k_{-m}(p_1y_1)k_{-n}(p_2y_2)f(p_1, p_2)\,dp_1\,dp_2$$

$$= 2^{(a-1)m}2^{(a'-1)n}\Gamma(am-m+1)\Gamma(a'n-n+1)y_1^{\mu_1+m}y_2^{\mu_2+n} \times$$

$$\times \int_0^\infty \int_0^\infty p_1^{\mu_1-(a-1)m}p_2^{\mu_2-(a'-1)n}f(p_1, p_2)\,dp_1\,dp_2 \times$$

$$\times \int_0^\infty \int_0^\infty \frac{x_1^{(a-2)m+1}J_{(a-2)m}(p_1x_1)x_2^{(a'-2)n+1}J_{(a'-2)n}(p_2x_2)\,dx_1\,dx_2}{(x_1^2+y_1^2)^{(a-1)m+1}(x_2^2+y_2^2)^{(a'-1)n+1}}$$

$$= 2^{(a-1)m}2^{(a'-1)n}\Gamma(am-m+1)\Gamma(a'n-n+1)y_1^{\mu_1+m}y_2^{\mu_2+n} \times$$

$$\times \int_0^\infty \int_0^\infty \frac{x_1^{(a-2)m+\frac{1}{2}}x_2^{(a'-2)n+\frac{1}{2}}\,dx_1\,dx_2}{(x_1^2+y_1^2)^{(a-1)m+1}(x_2^2+y_2^2)^{(a'-1)n+1}}$$

$$\times \int_0^\infty \int_0^\infty (p_1x_1)^{\frac{1}{2}}(p_2x_2)^{\frac{1}{2}}J_{(a-2)m}(p_1x_1)J_{(a'-2)n}(p_2x_2)p_1^{-\nu_1}p_2^{-\nu_2} \times$$

$$\times f(p_1, p_2)\,dp_1\,dp_2; \tag{36}$$

From (35) and (36) we get

$$\int_0^\infty \int_0^\infty \frac{x_1^{\mu_1-m}x_2^{\mu_1-n}f(x_1, x_2)\,dx_1\,dx_2}{(x_1^2+y_1^2)^{(a-1)m+1}(x_2^2+y_2^2)^{(a'-1)n+1}} = \int_0^\infty \int_0^\infty \frac{x_1^{\mu_1+\nu_1-m}x_2^{\mu_2+\nu_2-n}F(x_1, x_2)\,dx_1\,dx_2}{(x_1^2+y_1^2)^{(a-1)m+1}(x_2^2+y_2^2)^{(a'-1)n+1}}, \tag{37}$$

where

$$F(x_1, x_2) = \int_0^\infty \int_0^\infty (p_1x_1)^{\frac{1}{2}}(p_2x_2)^{\frac{1}{2}}J_{(a-2)m}(p_1x_1)J_{(a'-2)n}(p_2x_2)\,p_1^{-\nu_1}p_2^{-\nu_2}f(p_1, p_2)\,dp_1\,dp_2. \tag{38}$$

By applying Lemma 4 of the same paper referred to before, we get from (37)

$$F(x_1, x_2) = x_1^{-\nu_1}x^{-\nu_2}f(x_1, x_2), \tag{39}$$

provided the conditions required for the validity of the lemma as referred to above in the theorem are satisfied.

Theorem VIII. Conversely, let $x_1^{-\nu_1}x_2^{-\nu_2}f(x_1, x_2)$ be $R_{(a-2)m}$ and $R_{(a'-2)n}$ with respect to x_1 and x_2 respectively; then $x_1^{\nu_1}x_2^{\nu_2}\psi(x_1, x_2)$ is R_{am} and $R_{a'n}$ with respect to x_1 and x_2 respectively, where $\nu_1+\mu_1 = (a-1)m+\frac{1}{2}$ and $\nu_2+\mu_2 = (a'-1)n+\frac{1}{2}$, provided the integral for self-reciprocal property exists

Proceeding in the same way as Theorem VI, we can deduce this result from (35) and (36).

The following results established in the paper referred to are obtained as particular cases from theorems VII and VIII :

(a′) Theorem K and L, when $m = n = \frac{1}{2}$ and $\mu_1 = \mu_2 = \frac{1}{2}$, (*b*′) Theorems and analogous to M and N and which are extensions to two variables of Theorems V and VI (Mitra and Bose, 1953), when $m = n = \frac{1}{2}$, $\mu_1 = \mu_2 = \frac{1}{2}$ and $a = 1$, $a' = 1$,

(*c*′) Theorems O and P, when $\mu_1 = \mu_2 = \frac{1}{2}$

In conclusion I wish to express my thanks to Dr. S. C. Mitra for his kind help in the preparation of this paper.

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NEW FORMULATION OF EQUATIONS OF COMPRESSIBLE FLOW

By

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Introduction—The number of problems in compressible steady flow whose exact solutions have been obtained in two and three dimensions is not very large. An exact solution for the uniform motion of the sphere in a non-viscous gas has been recently obtained by J. R. Foote, (1950). He applies the "Synthetic Method" due to B. R. Seth, (1950). We propose to transform the equation for an irrotational flow in such a manner that its solution is as readily available as those of other equations of mathematical physics. This combined with Bernoulli's equation and the equation of state can give the complete solution. But the difficulty in all such cases is to find solutions common to both and satisfying a set of boundary conditions. If an external potential can be supposed to act on the field, the problem can readily be solved. For example, it is found that a heavy adiabatic gas, when gravity is taken into account, can flow uniformly in a region bounded by a plane provided the density varies as $y^{1/(\gamma-1)}$, y being the distance from the plane and γ, the adiabatic constant. But in general such an assumption is bound to be too artificial in character. If the motion is slow, the problem is again manageable; but its field of application becomes very limited.

1. We shall assume the motion to be irrotational and steady. If φ is the velocity potential and ϱ the density of the fluid, the equation of continuity is

$$\varrho\nabla^2\varphi + \nabla\varphi\nabla\varrho = 0. \tag{1.1}$$

Putting

$$\varphi_1 = \varrho^{\frac{1}{2}}\varphi, \quad \varrho_1 = \varrho^{\frac{1}{2}} \tag{1.2}$$

we get the equation of continuity in the canonical form

$$\frac{\nabla^2\varphi_1}{\varphi_1} = \frac{\nabla^2\varrho_1}{\varrho_1}, \tag{1.3}$$

or

$$\varrho_1\nabla^2\varphi_1 - \varphi_1\nabla^2\varrho_1 = 0.$$

This form does not seem to have been noticed before. It shows that

(*i*) it holds good in all dimensions;

(*ii*) it is an invariant in all systems of two-dimensional orthogonal curvilinear co-ordinates,

(*iii*) Its solutions can be obtained with greater ease. For example, if ϱ_1 is harmonic, so is φ_1; if ϱ_1 satisfies the wave equation so does φ_1, If $\gamma = 2$, and c is the local velocity of sound, equation (1.3) reduces to

$$\frac{\nabla^2(c\varphi)}{c\varphi} = \frac{\nabla^2 c}{c}. \tag{1.4}$$

In two dimensions we can obtain results in terms of the stream function as well. If ψ is the stream function, and we put

$$\psi_1 = \varrho^{-\frac{1}{2}}\psi, \quad \varrho_1 = \varrho^{-\frac{1}{2}}, \tag{1.5}$$

we get the same form of the equation of continuity as given in (1.3).

Thus, it appears that it is better to use the scalar quantities $\varrho^{\frac{1}{2}}\varphi$ and $\varrho^{\frac{1}{2}}$ in place of φ and ϱ.

2. When the variables are separable (1.3) can be solved in all systems of two dimensions orthogonal curvilinear co-ordinates. Let

$$z = f(\zeta) = f(\xi + i\eta). \tag{2.1}$$

Putting $$\varphi_1 = f_1(\xi)F_1(\eta), \quad \varrho_1 = f_2(\xi)\, F_2(\eta), \tag{2.2}$$

we see that the continuity equation (1.3) gives

$$\frac{f_1''}{f_1} + \frac{F_1''}{F_1} - \frac{f_2''}{f_2} - \frac{F_2''}{F_2} = 0, \tag{2.3}$$

where dashes denote differentiation with respect to ξ or η.

Thus, if n is a constant, we can put

$$\frac{f_1''}{f_1} - \frac{f_2''}{f_2} = n^2, \qquad \frac{F_1''}{F_1} - \frac{F_2''}{F_2''} = -n^2. \tag{2.3}$$

Assuming that $f_1 = f_2 \sinh \mu\xi$, $F_1 = F_2 \sin \lambda\eta$, we find that the density ϱ is given by

$$\varrho = B(\cosh \mu\, \xi)^{(n^2-\mu^2)/\mu^2} \cos (\lambda\eta)^{(n^2-\lambda^2)/\lambda^2}, \tag{2.4}$$

and φ by $$\varphi = A \sinh \mu\xi \sin \lambda\eta, \tag{2.5}$$

A and B being constants of integration, and μ, λ constants whose values have still to be determined.

If f_2 is a constant, the density is a function of η only, and f_1 becomes

$$f_1 = c_1 e^{n\xi} + c_2 e^{-n\xi}. \tag{2.6}$$

In the particular case of $n = 0$ the solutions take the form

$$\varphi = \left[c_1 + c_2 \int f_2^{-2} d\xi\right]\left[D_1 + D_2 \int F_2^{-2} d\eta\right],$$

$$\varrho = f_2(\xi)F_2(\eta). \tag{2.7}$$

A number of other solutions can similarly be obtained. As we have already noticed, if ϱ_1 is harmonic, so is φ_1. In such a case φ is the quotient of two harmonics. Again, if

$$\frac{\nabla^2(\varrho^{\frac{1}{2}}\varphi)}{\varrho^{\frac{1}{2}}\varphi} = \frac{\nabla^2 \varrho^{\frac{1}{2}}}{\varrho^{\frac{1}{2}}} = -k^2, \tag{2.8}$$

φ can be written as the quotient of solutions of the wave equation, k being a constant.

3. All solutions of (1.3) are to satisfy the equation

$$\int \frac{dp}{\varrho} + \tfrac{1}{2}q^2 + \Omega = C, \tag{3.1}$$

where q is the velocity, Ω the external potential and C a constant. For the adiabatic law

$$p = k\varrho^{\gamma}, \tag{3.2}$$

and equation (3.1) can be rewritten as

$$2k\gamma\varrho^{\gamma-1} = (\gamma-1)(\bar{q}^2-q^2)-2(\gamma-1)\Omega. \tag{3.3}$$

In general Ω is taken as zero, and hence

$$2k\gamma\varrho^{\gamma-1} = (\gamma-1)(\bar{q}^2-q^2). \tag{3.4}$$

Very few solutions of (1 3) are found to satisfy the equation (3.4). But they satisfy (3.3) if a suitable value of Ω is taken. Let us take the motion of a uniform gas bounded by a plane. In this case we put f_2 = a constant in (2.2), and $n = 0$ in (2.3). Thus we get

$$f_1 = A_1x+B_1, \quad F_1 = C_1+D_1\int F_2^{-2}\,dy, \tag{3.5}$$

$$\varphi = (A_1x+B_1)\left[C_1+D_1\int F_2^{-2}dy\right]. \tag{3.6}$$

If we substitute these values in (3.4) we find that $A_1 = 0$, F_2 and, hence ϱ is a constant. If we take gravity into consideration such that $\Omega = -gy$, we get

$$2k\gamma F_2^{2(\gamma-1)} = gy, \quad D_1 = 0, \quad A_1 = \bar{q}. \tag{3.7}$$

In this case $\varrho(=F_2^2)$ varies as $y^{1/(\gamma-1)}$. For $\gamma = \frac{3}{2}$ there is a parabolic distribution of density and the relation between p and ϱ is of the semi-cubical parabolic type. The flow is given by $\varphi = A_1x+B_1$ and its velocity q is uniform along the x-axis. The normal velocity, density and pressure all vanish on the x-axis, which can be taken as the bounding plane.

If polar co-ordinates are taken so that $z = \exp(\xi+i\eta)$, we can get the flow round a corner investigated by Prandtl.

In this case we take $n = 1$ in (2.6), put $c_2 = 0$, as thus get

$$\varphi = A\gamma \sin(\lambda\,\theta). \tag{3.8}$$

Similarly the simple cases of radial flow and flow in concentric circles can be obtained.

In three dimensions if $\varrho^{\frac{1}{2}}$ is harmonic and of the form $(1/r)$, we can take

$$\varphi_1 = -A\frac{Ua^2\cos\theta}{r^2}, \tag{3.9}$$

so that

$$\varphi = A_1\frac{\cos\theta}{r}. \tag{3.10}$$

This satisfies (3.4) if the pressure-density relation is

$$p = p_o - k\varrho^3. \tag{3.11}$$

This is the solution obtained by Foote (1950).

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ON SOME THEOREMS AND INEQUALITIES IN OPERATIONAL CALCULUS WITH TWO VARIABLES

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1. The present paper is a continuation of two other papers on operational calculus with two variables of the author. The derivation of some operational representations including some theorems on inequalities involving Laplace Transform of two variables together with some applications of the theorems established has been attempted. We shall use the notation

$$f(p, q) \doteqdot h(x, y)$$

to represent the convergent double integral

$$f(p, q) = pq \int_0^\infty \int_0^\infty e^{-px-qy} h(x, y)\, dx\, dy, \quad R(p) > 0, \ R(q) > 0.$$

Further, the equality

$$\int_0^\infty \int_0^\infty du\,dv \int_0^\infty \int_0^\infty \varphi(u, v, x, y) dx\, dy = \int_0^\infty \int_0^\infty dx\, dy \int_0^\infty \int_0^\infty \varphi(u, v, x, y) du\, dv, \qquad (A)$$

where

$$\varphi(u, v, x, y) = \varphi_1(u, v, x, y)\, \psi_1(x, y)$$
$$= \varphi_2(u, v, x, y)\, \psi_2(u, v) \qquad (B)$$

will be taken to hold when the following conditions are satisfied, *viz.*,

(*a*) $\varphi_1(u, v, x, y)$ is bounded and integrable in $(0 \leq x \leq \alpha,\ 0 \leq y \leq \beta)$ for every u, v in $0 \leq u \leq b_1$, $0 \leq v \leq b_2$, where α and β are arbitrary (*C*)

(*b*) $\int_0^\infty \int_0^\infty \psi_1(x, y)\, dx\, dy$ is absolutely convergent. (*D*)

(*c*) $\varphi_2(u, v, x\ y)$ is bounded and integrable in $(0 \leq u \leq \gamma,\ 0 \leq v \leq \delta)$ for every x, y in $0 \leq x \leq a_1$, $0 \leq y \leq a_2$, where γ and δ are arbitrary; (*E*)

(*d*) $\int_0^\infty \int_0^\infty \psi_2(x, y)\, dx\, dy$ is also absolutely convergent. (*F*)

In the theorems which follow, we shall only mention the conditons under which (*C*), (*D*), (*E*) and (*F*) hold and the changes in the orders of integrations etc effected in the theorems will be taken to be justified.

2. We have the operational representation

$$\frac{1}{p^\lambda q^\mu}\,{}_rE_s\left(\begin{matrix}\alpha_1,\ldots\ldots\alpha_r\\ \beta_1,\ldots\ldots\beta_s\end{matrix};\ \pm\frac{1}{pq}\right)$$

$$\doteqdot\frac{x^\lambda y^\mu}{\Gamma(\lambda+1)\Gamma(\mu+1)}\,{}_rF_{s+2}\left(\begin{matrix}\alpha_1,\ldots\ldots\alpha_r\\ \beta_1,\ldots\ldots\beta_s,\ \lambda+1,\ \mu+1\end{matrix};\ \pm xy\right) \tag{2.1}$$

Taking $r=1$, $s=0$, we obtain from (2 1)

$$\frac{p^{a-\lambda}q^{a-\mu}}{(1+pq)^a}\doteqdot\frac{x^\lambda y^\mu}{\Gamma(\lambda+1)\Gamma(\mu+1)}\,{}_1F_2\left(\begin{matrix}a\\ \lambda+1,\ \mu+1\end{matrix};\ -xy\right), \tag{2.2}$$

from which

$$\frac{p^{\frac12}q^{\frac12-\nu}}{1+pq}\doteqdot x^{-\frac12\nu}y^{\frac12\nu}H_\nu(2(xy)^{\frac12}), \tag{2.3}$$

since

$$H_\nu(z)=\frac{2(\frac12 z)^{\nu+1}}{\Gamma(\nu+\frac32)\pi^{\frac12}}\,{}_1F_2(1,\tfrac32,\nu+\tfrac32;\ -\tfrac14 z^2).$$

In particular,

$$(pq)^{\frac12}/(1+pq)\doteqdot H_0(2(xy)^{\frac12}). \tag{2.4}$$

Again from (2.1)

$$\frac{1}{p^\lambda q^\mu}\,{}_2F_1\left(\begin{matrix}\alpha_1,\ \alpha_2\\ \beta\end{matrix};\ -\frac{1}{pq}\right)\doteqdot\frac{x^\lambda y^\mu}{\Gamma(\lambda+1)\,\Gamma(\mu+1)}\,{}_2F_3\left(\begin{matrix}\alpha_1,\ \alpha_2\\ \beta,\ \lambda+1,\ \mu+1\end{matrix};\ -xy\right), \tag{2.5}$$

from which, we derive,

$$\frac{pq^{\lambda-\mu+1}}{(1+pq)^{\lambda+1}}\doteqdot\frac{x^{\lambda-\frac12\mu}y^{\frac12\mu}}{\Gamma(\lambda+1)}\,J_\mu(2(xy)^{\frac12}), \tag{2.6}$$

a result obtained by Delerue (1951) in his Thesis, p. 11 by an alternative method.

Also

$$\frac{pq}{1+pq}\doteqdot J_0(2(xy)^{\frac12}). \tag{2.7}$$

(Mclachlan and Humbert, 1941)

Further, from (2.1),

$$\frac{1}{p^{\lambda-\frac12}q^{\mu-\frac12}}\,{}_2F_2\left(\begin{matrix}\alpha_1,\ \alpha_2\\ \lambda+1,\ \mu+1\end{matrix};\ \frac{1}{pq}\right)\doteqdot\frac{x^{\lambda-\frac12}y^{\mu-\frac12}}{\Gamma(\lambda+\frac12)\,\Gamma(\mu+\frac12)}\,{}_2F_4\left(\begin{matrix}\alpha_1,\ \alpha_2\\ \lambda+\frac12,\ \lambda+1,\ \mu+\frac12,\ \mu+1\end{matrix};\ xy\right). \tag{2.8}$$

On further specialisation,

$$p^{-m}q^{-n}e^{1/pq}\doteqdot(-1)^{m+n}x^{\frac13 m-\frac13 n}y^{\frac13 n-\frac13 m}J_{m,n}(3(xy)^{\frac13}), \tag{2.9}$$

a result obtained by Delerue (1951) in his Thesis, (p. 11) by a different method;

$J_{m,n}(x)$ representing Humbert's function

$$J_{m,n}(x)=\frac{(\frac13 x)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}\,{}_0F_2(m+1,n+1,\ -x^3/27).$$

Since $\quad {}_1F_2(\alpha+\tfrac{1}{2}\,;\,\alpha+1,\ 2\alpha\,;\,-x^2) = \Gamma(\alpha)\Gamma(\alpha+1)(\tfrac{1}{2}x)^{1-2\alpha}J_\alpha(x)J_{\alpha-1}(x),\ R(\alpha)>0,$

$${}_1F_2(\tfrac{1}{2}\,;\,1+\alpha,\ 1-\alpha\,;\,-x^2) = \Gamma(1+\alpha)\Gamma(1-\alpha)J_\alpha(x)J_{-\alpha}(x),\quad -1<R(\alpha)<1,$$

and $\quad {}_1F_2(\alpha-\tfrac{1}{2}\,;\,\alpha,\ 2\alpha-1\,;\,-x^2) = \{\Gamma(\alpha)(\tfrac{1}{2}x)^{1-\alpha}J_{\alpha-1}(x)\}^2,\ R(\alpha)>0,$

we obtain, from (2.2), the following operational representations

$$\frac{p^{\frac{1}{2}}q^{3/2-\alpha}}{(1+pq)^{\alpha+\frac{1}{2}}} \doteqdot \frac{\pi^{\frac{1}{2}}}{\Gamma(\alpha+\frac{1}{2})}\, x^{\frac{1}{2}}y^{\alpha-\frac{1}{2}}J_\alpha((xy)^{\frac{1}{2}})J_{\alpha-1}((xy)^{\frac{1}{2}}) \tag{2.10}$$

$$\left(\frac{q}{p}\right)^{\alpha}\frac{(pq)^{\frac{1}{2}}}{(1+pq)^{\frac{1}{2}}} \doteqdot \left(\frac{x}{y}\right)^{\alpha}J_\alpha((xy)^{\frac{1}{2}})J_{-\alpha}((xy)^{\frac{1}{2}}) \tag{2.11}$$

and

$$\frac{p^{\frac{1}{2}}q^{\frac{1}{2}-\alpha}}{(1+pq)^{\alpha+\frac{1}{2}}} \doteqdot \frac{\pi^{\frac{1}{2}}}{\Gamma(\alpha+\frac{1}{2})}\, y^{\alpha}\{J_\alpha((xy)^{\frac{1}{2}})\}^2. \tag{2.12}$$

In Chakravarty's theorem (Chakravarty, 1953), *viz.*,

If $\varphi(u,v) \doteqdot f(x,y)$ and $g(x) \doteqdot \psi(u)$, then

$$\int_0^\infty \frac{\varphi(u,v)\,g(u)}{uv}\,du = \int_0^\infty\int_0^\infty \frac{f(x,y)\psi(x)}{x}\,e^{-vy}\,dx\,dy,$$

let us take $\quad f(x,y) = \dfrac{x^{\lambda-\mu}y^{\mu}}{\Gamma(\lambda+1)}J_{2\mu}(2(xy)^{\frac{1}{2}}),$

so that $\quad \varphi(u,v) = \dfrac{uv^{\lambda-2\mu+1}}{(1+uv)^{\lambda+1}}\,;$

and $g(x) = x^{-1-2\nu}e^{-1/x}$, so that

$$\psi(u) = 2u^{\nu+1}K_{2\nu}(2u^{\frac{1}{2}})\,;$$

[Mclachlan and Humbert, 1941 (p. 16)].

We then obtain on simplification, the integral

$$\int_0^\infty\int_0^\infty x^{\lambda-\mu+\nu}y^{\mu}J_{2\mu}(2(xy)^{\frac{1}{2}})K_{2\nu}(2x^{\frac{1}{2}})e^{-vy}\,dx\,dy$$

$$= \tfrac{1}{2}\Gamma(\lambda+1)\,\Gamma(\lambda+2\nu+1)\,v^{\alpha}e^{\frac{1}{2}v}\,W_{k,m}(v) \tag{2.13}$$

where $\alpha = \lambda+\nu-2\mu-\frac{1}{2},\ k = -\lambda-\nu-\frac{1}{2},\ m = \nu\,;$
the integral being valid for

$$\mu>-\tfrac{1}{2},\ \lambda-\mu>-\tfrac{1}{2},\ \lambda-\mu+2\nu>-\tfrac{1}{2}.$$

It may be noted that the above integral may also be evaluated by the application of Meijer's [Meijer, 1924] result

$$\int_0^\infty e^{-u^2}K_{2m}(2\xi u)u^{-2k}\,du = \frac{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}{4\xi}\,e^{\frac{1}{2}\xi^2}W_{k,m}(\xi^2),\ R(\tfrac{1}{2}\pm m-k)>0.$$

In particular, taking $\lambda = -\nu-\frac{1}{2}$, we obtain

$$\int_0^\infty\int_0^\infty x^{-\mu-\frac{1}{2}}y^{\mu}J_{2\mu}(2(xy)^{\frac{1}{2}})K_{2\nu}(2x^{\frac{1}{2}})e^{-vy}\,dx\,dy$$

$$= \frac{\Gamma(\frac{1}{2}-\nu)\Gamma(\frac{1}{2}+\nu)}{2\pi^{\frac{1}{2}}}\,v^{-2\mu-\frac{1}{2}}e^{\frac{1}{2}v}K_{\nu}(\tfrac{1}{2}v), \tag{2.14}$$

valid for $-2\mu\pm2\nu > -\frac{1}{2}$, $\mu > -\frac{1}{2}$, by analytic continuation.

Similarly, taking $f(x, y) = \dfrac{x^{\lambda-\frac{1}{2}}\,y^{\mu-\frac{1}{2}}}{\Gamma(\lambda+\frac{1}{2})\,\Gamma(\mu+\frac{1}{2})}\,{}_1F_2\left(\begin{matrix}\alpha\\ \lambda+\frac{1}{2},\ \mu+\frac{1}{2}\end{matrix};\ -xy\right)$

so that $\varphi(u, v) = \dfrac{u^{\alpha+\frac{1}{2}-\lambda}\,v^{\alpha+\frac{1}{2}-\mu}}{(1+uv)^{\alpha}}$, in Chakravarty's theorem mentioned above, we obtain

$$\int_0^\infty\int_0^\infty x^{\lambda+\nu-\frac{1}{2}}y^{\mu-\frac{1}{2}}\,{}_1F_2\left(\begin{matrix}\alpha\\ \lambda+\frac{1}{2},\ \mu+\frac{1}{2}\end{matrix};\ -xy\right)K_{2\nu}(2x^{\frac{1}{2}})e^{-vy}\,dx\,dy$$

$$= \tfrac{1}{2}\Gamma(\lambda+\tfrac{1}{2})\Gamma(\mu+\tfrac{1}{2})\Gamma(\lambda+2\nu+\tfrac{1}{2})e^{\frac{1}{2}v}v^{\delta}W_{\beta,\gamma}(v) \tag{2.15}$$

where $\delta = \frac{1}{2}(\alpha+\lambda+2\nu-2\mu-\frac{3}{2})$, $\beta = -\frac{1}{2}(\alpha+\lambda+2\nu-\frac{1}{2})$, $\gamma = \frac{1}{2}(\lambda+2\nu-\alpha+\frac{1}{2})$;

valid for $\mu > -\frac{1}{2}$, $\lambda+\nu\pm\nu > -\frac{1}{2}$.

We have the operational representation

$$x^{2\lambda}y^{2\mu}\,{}_0F_3(\beta_1, \beta_2, \beta_3;\ \pm x^2y^2)$$

$$\doteqdot \frac{\Gamma(2\lambda+1)\Gamma(2\mu+1)}{p^{2\mu}q^{2\mu}}\,{}_4F_3\left(\begin{matrix}\lambda+1,\ \mu+1,\ \lambda+\frac{1}{2},\ \mu+\frac{1}{2}\\ \beta_1,\ \beta_2,\ \beta_3\end{matrix};\ \pm\frac{16}{p^2\,q^2}\right). \tag{2.16}$$

The function $W_n^b(x)$ being defined by

$$W_n^b(x) = \mathrm{ber}_n(x)\ \mathrm{bei}_n'(x)-\mathrm{bei}_n(x)\ \mathrm{ber}_n'(x)$$

$$= \frac{x^{2n+1}}{2^{2n+1}n\,!\,(n+1)\,!}\,{}_0F_3\left(n+1,\ \tfrac{1}{2}n+1,\ \tfrac{1}{2}n+\tfrac{3}{2};\ \frac{x^4}{2^6}\right),$$

[Costello (1936) and Humbert (1937)],

it is easy to obtain from (2.16)

$$x^{2\lambda-n-\frac{1}{2}}\,y^{2\mu-n-\frac{1}{2}}\,W_n^b(2(2xy)^{\frac{1}{2}})$$

$$\doteqdot \frac{2^{n+\frac{1}{2}}\Gamma(2\lambda+1)\,\Gamma(2\mu+1)}{n\,!\ (n+1)\,!}\,\frac{1}{p^{2\lambda}q^{2\mu}}\,{}_4F_3\left(\begin{matrix}\lambda+1,\ \mu+1,\ \lambda+\frac{1}{2},\ \mu+\frac{1}{2}\\ n+1,\ \frac{1}{2}n+1,\ \frac{1}{2}n+\frac{3}{2}\end{matrix};\ \frac{16}{p^2\,q^2}\right) \tag{2.17}$$

In particular,

$$x^{-\frac{1}{2}}y^{\frac{1}{2}}W_n^b(2(2xy)^{\frac{1}{2}}) \doteqdot 2^{n+\frac{1}{2}}\frac{1}{p^nq^{n+1}}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}n+\frac{1}{2},\ \frac{1}{2}n+1\\ n+1\end{matrix};\ \frac{16}{p^2q^2}\right)$$

$$= (-1)^n2^{\frac{1}{2}-3n}p\,(p^2q^2-16)^{-\frac{1}{2}}((p^2q^2-16)^{\frac{1}{2}}-pq)^n \tag{2.18}$$

3. We write the theorem of P. Delerue [Delerue, p. 24] *viz.*,

'If $f(p, q) \doteqdot h(x, y)$ and $f(p^{\frac{1}{2}}, q^{\frac{1}{2}}) \doteqdot g(x, y)$, then

$$\pi (pq)^{\frac{1}{2}} g\left(\frac{1}{4p}, \frac{1}{4q}\right) \doteqdot \frac{1}{(xy)^{\frac{1}{2}}} h(x^{\frac{1}{2}}, y^{\frac{1}{2}})\text{'—in the form}$$

"If $f(p, q) \doteqdot h(x, y)$, $f(p^{\frac{1}{2}}, q^{\frac{1}{2}}) \doteqdot ((xy)^{\frac{1}{2}})\varphi\left(\frac{1}{x}, \frac{1}{y}\right)$, then

$$\varphi(p, q) \doteqdot \frac{1}{\pi (xy)^{\frac{1}{2}}} h(2x^{\frac{1}{2}}, 2y^{\frac{1}{2}}); \tag{3A}$$

valid for

$$\begin{aligned} h(x, y) &= 0(x^{\mu_1}); \text{ for small } x, \ R(\mu_1) > 0 \\ &= 0(y^{\mu_2}), \text{ for small } y, \ R(\mu_2) > 0 \end{aligned}$$

and (if necessary),

$$\begin{aligned} h(x, y) &= O\{\exp(-x^{\lambda_1})\}, \text{ for large } x, \quad R(\lambda_1) > 0 \\ &= O\{\exp(-y^{\lambda_2})\}, \text{ for large } y, \quad R(\lambda_2) > 0, \end{aligned}$$

and utilise the same in obtaining some new operational representations given below.

(1) Let $f(p, q) = \dfrac{1}{p^{\lambda}} \dfrac{p^2 q^2}{1 + p^2 q^2}$ so that

$$h(x, y) = (x/y)^{\lambda} \operatorname{ber}_{2\lambda}(2(xy)^{\frac{1}{2}}). \quad \text{[Delerue, (1951)—p. 59.]}$$

Since

$$f(p^{\frac{1}{2}}, q^{\frac{1}{2}}) = \frac{1}{p^{\lambda}} \frac{pq}{1 + pq} \doteqdot \left(\frac{x}{y}\right)^{\frac{1}{2}\lambda} J_{\lambda}(2(xy)^{\frac{1}{2}}),$$

[Delerue, 1951, p. 59], we obtain from the theorem (3.*A*) the following operational representation

$$p^{\frac{1}{2}(1-\lambda)} q^{\frac{1}{2}(1+\lambda)} J_{\lambda}\left(\frac{2}{(pq)^{\frac{1}{2}}}\right) \doteqdot \frac{\operatorname{ber}_{2\lambda}(4(xy)^{\frac{1}{2}})}{\pi\, x^{\frac{1}{2}(1-\lambda)} y^{\frac{1}{2}(1+\lambda)}}. \tag{3.1}$$

In particular

$$\frac{1}{\pi (xy)^{\frac{1}{2}}} \operatorname{ber}(4(xy)^{\frac{1}{2}}) \doteqdot ((pq)^{\frac{1}{2}}) J_0(2/(pq)^{\frac{1}{2}}) \tag{3.2}$$

a result obtained by Delurue in his thesis (Delurue, 1951).

(2) Similarly from

$$\left(\frac{x}{y}\right)^{\lambda} \operatorname{bei}_{2\lambda}(2(xy)^{\frac{1}{2}}) \doteqdot \frac{1}{p^{2\lambda}} \frac{pq}{1 + p^2 q^2}$$

[Delerue, 1951, p. 59]

and

$$\frac{1}{p^{\lambda}} \cdot \frac{(pq)^{\frac{1}{2}}}{1 + pq} \doteqdot \left(\frac{x}{y}\right)^{\frac{1}{2}\lambda} \mathrm{H}_{\lambda}(2(xy)^{\frac{1}{2}})$$

we obtain the operational representation

$$p^{\frac{1}{2}(1-\lambda)} q^{\frac{1}{2}(1+\lambda)} H_{\lambda}\left(\frac{2}{(pq)^{\frac{1}{2}}}\right) \doteqdot \frac{\operatorname{bei}_{2\lambda}(4(xy)^{\frac{1}{2}})}{\pi\, x^{\frac{1}{2}(1-\lambda)} y^{\frac{1}{2}(1+\lambda)}} \tag{3.3}$$

and in particular,

$$\frac{1}{\pi(xy)^{\frac12}}\operatorname{bei}(4(xy)^{\frac14}) \doteqdot (pq)^{\frac12}H_0(2/(pq)^{\frac12}) \tag{3.4}$$

(3) We have the operational representation

$$x^{\frac23 m-\frac13 n}\,y^{\frac23 n-\frac13 m}J_{m,n}(3(xy)^{\frac13}) \doteqdot \frac{1}{p^m q^n}e^{-1/pq}$$

[Delerue, 1951, p. 11], from which

$$\frac{1}{(pq)^{\frac13}}e^{-1/(pq)} \doteqdot (xy)^{\frac13}J_{\frac13,\frac13}(3(xy)^{\frac13}),$$

also

$$\frac{1}{p^2\,q^2}e^{-1/p^2q^2} \doteqdot xy\ {}_0F_4\!\left(1,\ 1,\ \tfrac32,\ \tfrac32;\ -\frac{x^2y^2}{16}\right),$$

obtained by Chakravarty in another paper.

Hence the theorem gives the following new operational representation

$$(pq)^{\frac13}J_{\frac13,\frac13}(3/(pq)^{\frac13}) \doteqdot (4/\pi)\,{}_0F_4(1,\ 1,\ \tfrac32,\ \tfrac32;\ -xy) \tag{3 5}$$

(4) From the operational representations

$$I_0(2(xy)^{\frac12}) \doteqdot \frac{pq}{pq-1} \text{ and}$$

$\frac12\{J_0(2(xy)^{\frac14})+I_0(2(xy)^{\frac14})\} \doteqdot \dfrac{p^2q^2}{p^2q^2-1}$ we obtain, by the theorem, the representation

$$(pq)^{\frac12}I_0(2/(pq)^{\frac12}) \doteqdot \frac{1}{2\pi(xy)^{\frac12}}[J_0(4(xy)^{\frac14})+I_0(4(xy)^{\frac14})].$$

(5) Finally, from the operational representations

$$\frac{(pq)^{\frac12}}{(1+pq)^{\frac12}} \doteqdot \{J_0(xy)^{\frac12}\}^2$$

and $\dfrac{pq}{(1+p^2q^2)^{\frac12}} \doteqdot I_0((2xy)^{\frac12})J_0((2xy)^{\frac12})$, we obtain, by the theorem, the operational relation

$$(pq)^{\frac12}\{J_0((pq)^{-\frac12}/8)\}^2 \doteqdot (\pi(xy)^{\frac12})^{-1}\,I_0((xy)^{\frac14})\,J_0((xy)^{\frac14}). \tag{3.7}$$

4. **Theorem I.*** *If* $\qquad h(x,\ y) \doteqdot f(p,\ q) \qquad (A)$

and $\qquad h(x^2,\ y^2) \doteqdot (4/(pq))\,\varphi(\tfrac14p^2,\ \tfrac14q^2), \qquad (B)$

then $\qquad \varphi(p,\ q) \doteqdot (\pi(xy)^{\frac12})^{-1}\,f(1/x,\ 1/y) \qquad (C)$

From $f(p,\ q) \doteqdot h(x,\ y)$,

$$h(x^2,\ y^2) \doteqdot \frac{pq}{\pi}\int_0^\infty\int_0^\infty e^{-\frac14p^2s^2-\frac14q^2t^2}f(1/s^2,\ 1/t^2)\,ds\,dt, \tag{4.1}$$

---

* This theorem is in essence different from that obtained by Delerue in his thesis p. 34., cor. 2. He has assumed there that (*A*) and (*C*) are true and established that (*B*) is then true.

Thus $$\frac{4}{p^2q^2}\varphi(\tfrac{1}{4}p^2, \tfrac{1}{4}q^2) = \frac{1}{\pi}\int_0^\infty\int_0^\infty e^{-\frac{1}{4}p^2s^2-\frac{1}{4}q^2t^2} f(1/s^2, 1/t^2)\, ds\, dt. \tag{4.2}$$

From (4.2), the the theorem follows on slight reduction.

As regards the validity of the theorem, we assume

$$\begin{aligned} h(x, y) &= 0(x^{\mu_1}), \text{ for small } x, \quad R(\mu_1) > 0 \\ &= 0(y^{\mu_2}), \text{ for small } y, \quad R(\mu_2) > 0 \\ h(x, y) &= O\{\exp(-x^{\lambda_1})\}, \text{ for large } x, \quad R(\lambda_1) > 0 \\ &= O\{\exp(-y^{\lambda_2})\}, \text{ for large } y, \quad R(\lambda_2) > 0. \end{aligned}$$

Cor. I. Making use of the operational representation

$$p^{-m}q^{-n}e^{-(1/pq)} \doteqdot x^{\frac{1}{3}(2m-n)}y^{\frac{1}{3}(2n-m)}J_{m,n}(3(xy)^{\frac{1}{3}}) \tag{4.2a}$$

in the formula (4.1), we obtain, after considerable simplification

$$x^{\frac{1}{3}(4m-2n)}y^{\frac{1}{3}(4n-2m)}J_{m,n}(3(x^2y^2)^{\frac{1}{3}})$$
$$\doteqdot \frac{\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\pi\, p^{m-n}q^{n-m}} e^{p^2q^2/32} W_{-\frac{1}{2}(m+n), \frac{1}{2}(m-n)}(p^2q^2/16) \tag{4.3}$$

In particular

$$x^{2m}y^{-2m}J_{m,-m}(3(x^2y^2)^{\frac{1}{3}} \doteqdot \frac{\Gamma(\frac{1}{2}+m)\Gamma(\frac{1}{2}-m)}{4n^{3/2}} p^{1-2m}q^{1+2m}e^{p^2q^2/32}K_m(p^2q^2/32), \tag{4.4}$$

and $$J_{o,o}(3(x^2y^2)^{\frac{1}{3}}) \doteqdot \frac{pq}{4\pi^{\frac{1}{2}}} e^{p^2q^2/32}K_o(p^2q^2/32). \tag{4.5}$$

We now use the theorem with the operational representations (4.2*a*) and (4.8), so as to obtain $\varphi(p, q)$ of the theorem, given by

$$\varphi(p, q) = \frac{\Gamma(m+\frac{1}{2})\,\Gamma(n+\frac{1}{2})}{\pi\, p^{\frac{1}{2}(m-n-1)}\, q^{\frac{1}{2}(n-m-1)}} e^{\frac{1}{2}pq} W_{-\frac{1}{2}(m+n)\ \frac{1}{2}(m-n)}(pq).$$

We then obtain a new operational representation

$$x^{m-\frac{1}{2}}y^{n-\frac{1}{2}}e^{-xy} \doteqdot \frac{\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{p^{\frac{1}{2}(m-n-1)}q^{\frac{1}{2}(n-m-1)}}, e^{\frac{1}{2}pq}W_{-\frac{1}{2}(m+n), \frac{1}{2}(m-n)}(pq). \tag{4.6}$$

Since $W_{o,m}(x) = \left(\frac{x}{\pi}\right)^{\frac{1}{2}}K_m(\frac{1}{2}x)$, we obtain

$$\frac{e^{-xy}}{(xy)^{\frac{1}{2}}} \doteqdot \pi^{\frac{1}{2}}\, pq\, e^{\frac{1}{2}pq}K_o(\tfrac{1}{2}pq), \tag{4.7}$$

a result obtained by Delerue in his thesis (p 12) in a different manner;

and $$x^{m-\frac{1}{2}}y^{-m-\frac{1}{2}}e^{-xy} \doteqdot \frac{\Gamma(\frac{1}{2}+m)\,\Gamma(\frac{1}{2}-m)}{\pi^{\frac{1}{2}}} p^{1-m}q^{1+m}e^{\frac{1}{2}pq}K_m(\tfrac{1}{2}pq). \tag{4.8}$$

Since $E_i(x) \equiv \int_x^\infty \frac{e^{-x}}{x}\,dx = x^{-\frac{1}{2}}e^{-\frac{1}{2}x}W_{-\frac{1}{2},0}(x)$*

[*vide*, Whittaker and Watson, 1940, p. 352],

we obtain, from (4.6)

$$e^{-xy} \doteqdot pq\, e^{pq} E_i(pq). \tag{4.9}$$

a similar result with Humbert and Mclachlan's definition of $E_i(x)$ has already been given by Mclachlan and Humbert (1941).

Finally, $$y^{\nu-1}e^{-xy} \doteqdot \Gamma(\nu)\, pq^{2-\nu}s(\nu, pq)^{**} \tag{4.10}$$

where $s(\nu, z)$ is Schlömilch's function

$$s(\nu, z) = z^{\nu-1}e^{z}\int_0^\infty \frac{e^{-u}}{u^\nu}\,du = z^{\frac{1}{2}\nu-1}e^{\frac{1}{2}z}W_{-\frac{1}{2}\nu,\frac{1}{2}-\frac{1}{2}\nu}(z)$$

[Whittaker and Watson, 1940, p. 352].

Cor. II. Let $h(x, y) \equiv \log xy$; so that $f(p, q) \equiv -2\gamma-\log pq$

[Mclachlan & Humbert, 1941].

Then $$h(x^2, y^2) = 2\log xy \doteqdot -4\gamma-2\log pq.$$

Hence the theorem leads, on simplification, to the operational representation

$$(pq)^{\frac{1}{2}}(2\gamma-\log_e pq) \doteqdot \frac{1}{\pi(xy)^{\frac{1}{2}}}\{4(\gamma+\log_e 2)+\log_e xy\},$$

a result obtained by Delerue [Delerue, 1951, p. 25] in a different manner.

5. Theorem II. *Let $l>1$ and $\frac{1}{l}+\frac{1}{l'}=1$ and*

$$\int_0^\infty\int_0^\infty k(x, y)(xy)^{s-1}dx\,dy = \varphi(s),$$

then†

(*A*) $$\int_0^\infty\int_0^\infty f(x, y)\,dx\,dy\int_0^\infty\int_0^\infty k(xs, yt)g(s, t)\,ds\,dt$$

$$<\varphi\left(\frac{1}{l}\right)\left\{\int_0^\infty\int_0^\infty (xy)^{l-2}f^l(x, y)\,dx\,dy\right\}^{1/l}\times\left\{\int_0^\infty\int_0^\infty g^{l'}(x, y)\,dx\,dy\right\}^{1/l'}$$

---

* This definition is slightly different from that of Humbert and Mclachlan whose definition is

$$E_i(t) = \int_0^t \frac{e^x}{x}\,dx$$

** This result occurs in Voelker and Doetsch (1950)

† The functions involved in this section under the signs of integrations are assumed to be integrable with respect to each of the variables

$$(B)\quad \int_0^\infty\int_0^\infty ds\,dt\left\{\int_0^\infty\int_0^\infty k(xs,\,yt)\,f(x,\,y)\,dx\,dy\right\}^l$$

$$< \varphi^l\left(\frac{1}{l}\right)\int_0^\infty\int_0^\infty (xy)^{l-2}f^l(x,\,y)\,dx\,dy$$

$$(C)\quad \int_0^\infty\int_0^\infty (st)^{l-2}\,ds\,dt\left\{\int_0^\infty\int_0^\infty k(xs,\,yt)f(x,\,y)\,dx\,dy\right\}^l$$

$$< \varphi^l\left(\frac{1}{l'}\right)\int_0^\infty\int_0^\infty f^l(x,\,y)\,dx\,dy.$$

We have

$$\int_0^\infty\int_0^\infty f(x,\,y)\,dx\,dy\int_0^\infty\int_0^\infty k(xs,\,yt)\,g(s,\,t)\,ds\,dt$$

$$= \int_0^\infty\int_0^\infty f(x,\,y)\,dx\,dy\int_0^\infty\int_0^\infty k(u,\,v)\left(\frac{u}{x},\,\frac{v}{y}\right)\frac{du\,dv}{xy}$$

$$= \int_0^\infty\int_0^\infty k(u,\,v)\,du\,dv\int_0^\infty\int_0^\infty f(x,\,y)\,g\left(\frac{u}{x},\,\frac{v}{y}\right)\frac{dx\,dy}{xy} \tag{5.1}$$

$$= \int_0^\infty\int_0^\infty k(u,\,v)\,du\,dv\int_0^\infty\int_0^\infty \frac{1}{xy}f\left(\frac{1}{x},\,\frac{1}{y}\right)g(ux,\,vy)\,dx\,dy$$

$$< \int_0^\infty\int_0^\infty k(u,\,v)\,du\,dv\left\{\int_0^\infty\int_0^\infty (xy)^{-l}f^l\left(\frac{1}{x},\,\frac{1}{y}\right)dx\,dy\right\}^{1/l}$$

$$\times\left\{\int_0^\infty\int_0^\infty g^{l'}(ux,\,vy)\,dx\,dy\right\}^{1/l'}$$

[by Hölder's inequality extended to double integrals—*vide*. Hardy, Littlewood, Polya, 1934. p. 140]

$$= \int_0^\infty\int_0^\infty (uv)^{-1/l'}k(u,\,v)\,du\,dv\left\{\int_0^\infty\int_0^\infty (xy)^{l-2}f^l(x,\,y)\,dx\,dy\right\}^{1/l}$$

$$\times\left\{\int_0^\infty\int_0^\infty g^{l'}(x\,y)\,dx\,dy\right\}^{1/l'}$$

$$= \varphi\left(\frac{1}{l}\right)\left\{\int_0^\infty\int_0^\infty (xy)^{l-2}f^l(x,\,y)\,dx\,dy\right\}^{1/l}\times\left\{\int_0^\infty\int_0^\infty g^{l'}(x,\,y)\,dx\,dy\right\}^{1/l'} \tag{5.2}$$

which is (A).

As regards the changes in the order of integration in (5.1), we assume, in order that (*C*), (*D*), (*E*) and (*F*) of Art. 1 are satisfied, the following conditions

(*i*) each of $|k(u, v)|$, $|f(x, y)|$, $\left|\dfrac{1}{xy}g\left(\dfrac{u}{x}, \dfrac{v}{y}\right)\right| < \infty$ for $x \geq 0$, $y \geq 0$, $u \geq 0$, $v \geq 0$.

(*ii*) the integrals $\displaystyle\int_0^\infty\int_0^\infty |f(x, y)|\, dx\, dy$ and $\displaystyle\int_0^\infty\int_0^\infty |k(u, v)|\, du\, dv$ are convergent.

By an obvious extension to double integrals of a theorem, *viz.*,

"If $k > 1$, then a necessary and sufficient condition that $\int f^k dx \leq F$ is that $\int fg\, dx \leq F^{1/k} G^{1/k'}$ for all g such that $\int g^{k'} dx \leq G$." [Hardy, Littlewood & Polya, 1934, p. 142].

(*B*) follows directly from (*A*)

In (*B*) change $k(x, y)$ to $(xy)^{2/l'-1} k(x, y)$, and $f(x, y)$ to $(xy)^{2/l-1} f(x, y)$; and note that the consequent change in $\varphi(s)$ is $\varphi(2/l'+s-1)$. Then on slight simplification (*C* follows from (*B*).

We next apply the above theorems to obtain several inequalities in operational calculus with two variables. In what follows we assume all the double integrals involved as convergent and $1/l+1/l' = 1$, $l > 1$.

Let $k(x, y) = e^{-x-y} x^m y^n$, then

$$\varphi(s) = \Gamma(m+s)\,\Gamma(n+s). \tag{5 3}$$

Further, let

$$\psi(p, q) \doteqdot x^m y^n g(x, y)$$

and

$$\lambda(p, q) \doteqdot x^m y^n f(x, y).$$

Then

$$\int_0^\infty\int_0^\infty k(xs.\ yt)\, g(s, t)\, ds\, dt = x^{m-1} y^{n-1}\, \psi(x, y)$$

and

$$\int_0^\infty\int_0^\infty k(xs, yt) f(s, t)\, ds\, dt = x^{m-1} y^{n-1} \lambda(x, y).$$

By Goldstein's Theorem

$$\int_0^\infty\int_0^\infty x^{m-1} y^{n-1} f(x, y)\, \psi(x, y)\, dx\, dy = \int_0^\infty\int_0^\infty x^{m-1} y^{n-1} \lambda(x, y) g(x, y)\, dx\, dy.$$

Then from (A)

$$\int_0^\infty\int_0^\infty x^{m-1}y^{n-1}f(x,y)\psi(x,y)\,dx\,dy=\int_0^\infty\int_0^\infty x^{m-1}y^{n-1}\lambda(x,y)g(x,y)\,dx\,dy$$

$$<\Gamma(m+1/l)\,\Gamma(n+1/l)\left\{\int_0^\infty\int_0^\infty (xy)^{l-2}f^l(x,y)\,dx\,dy\right\}^{1/l}$$

$$\times\left\{\int_0^\infty\int_0^\infty g^{l'}(x,y)\,dx\,dy\right\}^{1/l'} \tag{5.4}$$

In particular, let us put $m=n=(l-l')/ll'$ in (5.4) and change $f(x,y)$ to $(xy)^{2/l-1}f(x,y)$; we then obtain:

If $\psi(p,q)\doteqdot(xy)^{(l-l')/ll'}g(x,y)$, then

$$\int_0^\infty\int_0^\infty \frac{f(x,y)\,\psi(x,y)}{xy}\,dx\,dy$$

$$<\{\Gamma(1/l')\}^2\left\{\int_0^\infty\int_0^\infty f^l(x,y)\,dx\,dy\right\}^{1/l}\times\left\{\int_0^\infty\int_0^\infty g^{l'}(x,y)\,dx\,dy\right\}^{1/l'} \tag{5.5}$$

Taking $l=l'=2$ and $f(x,y)=e^{-x-y}$, (5.5) reduces to

$$\int_0^\infty\int_0^\infty \frac{e^{-x-y}\psi(x,y)}{xy}\,dx\,dy<\tfrac{1}{2}\pi\left\{\int_0^\infty\int_0^\infty g^2(x,y)\,dx\,dy\right\}^{\frac{1}{2}} \tag{5.6}$$

From the operational representation (4.9), we obtain

$$e^{-xy-x-y}\doteqdot pq\;e^{(p+1)(q+1)}E_1\{(p+1)(q+1)\}^{*}$$

and so (5.6) leads to the inequality

$$\int_1^\infty\int_1^\infty {}^{xy-x-y}E_i(xy)\,dx\,dy<\frac{\pi}{4e^2} \tag{5.7}$$

where we have utilised the fact that

$$\int_0^\infty\frac{e^{-2x}dx}{1+x}<\tfrac{1}{2}. \tag{5.7a}$$

[The author took the left side of (5.7a) less than 1, but Prof. Delerue points out that this is less than $\frac{1}{2}$].

Again, if $\psi(p,q)\doteqdot g(x,y)$, $\lambda(p,q)\doteqdot f(x,y)$,

then
$$\int_0^\infty\int_0^\infty\frac{f(x,y)\,\psi(x,y)}{xy}\,dx\,dy=\int_0^\infty\int_0^\infty\frac{\lambda(x,y)\,g(x,y)}{xy}\,dx\,dy$$

$$<\{\Gamma(1/l)\}^2\left\{\int_0^\infty\int_0^\infty (xy)^{l-2}f^l(x,y)\,dx\,dy\right\}^{1/l}\left\{\int_0^\infty\int_0^\infty g^{l'}(x,y)\,dx\,dy\right\}^{1/l'}. \tag{5.8}$$

---

[*] $E_1(x)$ has the same definition as before and is slightly different from that of Mclachlin and Humbert.

Form (B),

If $\lambda(p,\, q) \doteqdot x^m y^n f(x,\, y)$, then

$$\int_0^\infty \int_0^\infty s^{(m-1)l} t^{(n-1)l} \lambda^l(s,\, t)\, ds\, dt$$

$$< \{\Gamma(m+1/l)\, \Gamma(n+1/l)\}^l \int_0^\infty \int_0^\infty (xy)^{l-2} f^l(x,\, y)\, dx\, dy. \tag{5.9}$$

In (5.9) let us take $m = n = 1/l'$ and replace $f(x,\, y)$ by $(xy)^{-1/l'} f(x,\, y)$; we then obtain

If $\lambda(p,\, q) \doteqdot f(x,\, y)$, then

$$\int_0^\infty \int_0^\infty \frac{\lambda^l(x,\, y)}{xy}\, dx\, dy < \int_0^\infty \int_0^\infty \frac{f^l(x,\, y)}{xy}\, dx\, dy\,; \tag{5.10}$$

and from (5.9) by simply taking $m = n = 0$, we have

$$\int_0^\infty \int_0^\infty \left\{\frac{\lambda(x,\, y)}{xy}\right\}^l dx\, dy < \{\Gamma(1/l)\}^{2l} \int_0^\infty \int_0^\infty (xy)^{l-2} f^l(x,\, y)\, dx\, dy \tag{5.11}$$

when $f(x,\, y) \doteqdot \lambda(p,\, q)$.

Further from (C),

If $\lambda(p,\, q) \doteqdot x^m y^n f(x,\, y)$, then

$$\int_0^\infty \int_0^\infty x^{ml-2} y^{nl-2} \lambda^l(x,\, y)\, dx\, dy$$

$$< \{\Gamma(m+1/l')\, \Gamma(n+1/l')\}^l \int_0^\infty \int_0^\infty f^l(x,\, y)\, dx\, dy \tag{5.12}$$

which may be put into the form

$$\int_0^\infty \int_0^\infty \lambda^l(x,\, y)\, dx\, dy < \{\Gamma(1+1/l)\}^{2l} \int_0^\infty \int_0^\infty f^l(x,\, y)\, dx\, dy$$

provided $\lambda(p,q) \doteqdot (xy)^{2/l} f(x,\, y)$, $l > 1$ and $1/l + 1/l' = 1$.

Putting $l = l' = 2$ and assuming further that

$$\begin{aligned} &f(x,\, y) \doteqdot \lambda(p,\, q) & & & g(x,\, y) \doteqdot \psi(p,\, q) \\ & & \text{and} & & \\ &pq\, f(p,\, q) \doteqdot v(x,\, y) & & & pq\, g(p,\, q) \doteqdot u(x,\, y). \end{aligned}$$

we obtain by Goldstein's Theorem

$$\int_0^\infty \int_0^\infty \frac{f(x,\, y)\psi(x,\, y)}{xy}\, dx\, dy = \int_0^\infty \int_0^\infty \frac{\lambda(x,\, y)\, g(x,\, y)}{xy}\, dx\, dy$$

$$\int_0^\infty \int_0^\infty f^2(x,\, y)\, dx\, dy = \int_0^\infty \int_0^\infty \frac{\lambda(x,\, y)\, v(x,\, y)}{xy}\, dx\, dy$$

and
$$\int_0^\infty \int_0^\infty g^2(x, y)\, dx\, dy = \int_0^\infty \int_0^\infty \frac{\psi(x, y)\, u(x, y)}{xy}\, dx\, dy$$

and the results (5.8), (5.11) and (5.13) take up the following forms:

$$\left\{\int_0^\infty \int_0^\infty \frac{f(x, y)\, \psi(x, y)}{xy}\, dx\, dy\right\}^2 = \left\{\int_0^\infty \int_0^\infty \frac{\lambda(x, y)\, g(x, y)}{xy}\, dx\, dy\right\}^2$$

$$< \pi^2 \left\{\int_0^\infty \int_0^\infty \frac{\lambda(x, y)\, v(x, y)}{xy}\, dx\, dy\right\}\left\{\int_0^\infty \int_0^\infty \frac{\psi(x, y)\, u(x, y)}{xy}\, dx\, dy\right\} \tag{5.14}$$

$$\int_0^\infty \int_0^\infty \left\{\frac{\lambda(x, y)}{xy}\right\}^2 dx\, dy < \pi^2 \int_0^\infty \int_0^\infty \frac{\lambda(x, y)\, v(x, y)}{xy}\, dx\, dy \tag{5.15}$$

and
$$\int_0^\infty \int_0^\infty \lambda^2(x, y)\, dx\, dy < \tfrac{1}{16}\pi^2 \int_0^\infty \int_0^\infty \frac{\lambda(x, y)\, v(x, y)}{xy}\, dx\, dy \tag{5.16}$$

6. Theorem III. *If* $\varphi(p, q) \doteqdot h(x, y)$ *and* $1 < l \leq 2$ *and* $1/l + 1/l' = 1$, *then*

$$\int_1^\infty \int_1^\infty \frac{\varphi^l(x, y)}{x^2 y^2}\, dx\, dy < \Gamma^{2l}(1/l') \int_0^\infty \int_0^\infty e^{-u-v} h(u, v)\}^l\, du\, dv,$$

the integrals involved being supposed convergent.

From $\varphi(p, q) \doteqdot h(x, y)$, we obtain

$$\varphi\left(\frac{1}{x}, \frac{1}{y}\right) = \frac{1}{xy} \int_0^\infty \int_0^\infty e^{-u/x - v/y} h(u, v)\, du\, dv \tag{6.1}$$

from which we obtain

$$\int_0^1 \int_0^1 \varphi^l\left(\frac{1}{x}, \frac{1}{y}\right) dx\, dy = \int_0^1 \int_0^1 \frac{dx\, dy}{(xy)^l} \left\{\int_0^\infty \int_0^\infty e^{-u/x - v/y} h(u, v)\, du\, dv\right\}^l.$$

On changing the variables x, y first to $1/x$, $1/y$ on both sides and then to $x+1$, $y+1$, on the right side only, we obtain ultimately

$$\int_1^\infty \int_1^\infty \frac{\varphi^l(x, y)}{x^2 y^3}\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty \{(x+1)(y+1)\}^{l-2} dx\, dy \left\{\int_0^\infty \int_0^\infty e^{-ux-vy}[e^{-u-v} h(u, v)]\, du\, dv\right\}^l \tag{6.2}$$

$$\leq \int_0^\infty \int_0^\infty (xy)^{l-2}\, dx\, dy \left\{\int_0^\infty \int_0^\infty e^{-ux-vy}[e^{-u-v} h(u, v)]\, du\, dv\right\}^l \tag{6.3}$$

as $1 < l \leq 2$.

The theorem now follows from an application of Theorem II (*C*).

As an application of the theorem, we start with the operational representation

$$p^{-m}q^{-n}e^{-1/pq} \doteqdot x^{\frac{2}{3}m-\frac{1}{3}n}y^{\frac{2}{3}n-\frac{1}{3}m}J_{m,n}(3(xy)^{\frac{1}{3}}).$$

Then after some reductions

$$\int_1^\infty\int_1^\infty \frac{\varphi^l(x,y)}{x^2y^2}\,dx\,dy = \frac{1}{l^{ml+1}}\int_0^l z^{ml-nl-1}\gamma(nl+1,z)\,dz \tag{6.4}$$

where $\gamma(n, x)$ is the Incomplete Gamma function defined by

$$\gamma(n,x) = \int_0^x e^{-t}t^{n-1}dt = \Gamma(n) - x^{\frac{1}{2}(n-1)}e^{-\frac{1}{2}x}W_{\frac{1}{2}(n-1),\frac{1}{2}n}(x).$$

We thus obtain the inequality

$$\int_0^\infty\int_0^\infty \{e^{-u-v}u^{\frac{1}{3}(2m-n)}v^{\frac{1}{3}(2n-m)}J_{m,n}(3(uv)^{\frac{1}{3}})\}^l\,du\,dv$$

$$> \frac{1}{l^{ml+1}\Gamma^{2l}(1/l')}\int_0^\infty z^{ml-nl-1}\gamma(nl+1,z)\,dz \tag{6.5}$$

Putting $l = 2$, $m = \frac{1}{2}$ and $n = 0$ and observing that

$$J_{\frac{1}{2},0}(3(xy)^{\frac{1}{3}}) = (2/\pi^{\frac{1}{2}})(xy)^{\frac{1}{2}}J_1{}^2(4xy)$$

where $J_\gamma{}^\mu(x)$ is Wright's generalised Bessel function

$$J_\gamma^\mu(x) = \sum_{r=0}^\infty \frac{(-x)^r}{r!\,\Gamma(1+\lambda+\mu r)},\ \mu > 0,$$

[Wright E. M.—1934], in which

$$J_\gamma^\mu(x) \sim O\left[x^{-k(\frac{1}{2}+\lambda)}\exp\left\{\frac{(\mu x)^k}{\mu k}\cos\pi k\right\}\right] \text{ for large } x,\ k = \frac{1}{\mu+1}$$

$$\sim O(1), \text{ for small } x,$$

we obtain

$$\int_0^\infty\int_0^\infty e^{-2u-2v}u\{J_1{}^2(4uv)\}^2\,du\,dv > 1/16\pi(1+1/e^2) \tag{6.6}$$

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SHORTEST DISTANCE *

By

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(*Received—January 1, 1952.*)

Let a', b', c', be the shortest distances of the pairs of lines b, c; c, a; a, b and a'', b'', c'' be the shortest distances of the pairs of lines a, a'; b, b'; c, c'; then the triad of lines a'', b'', c'' have the same shortest distance d. (Peterson-Morley Theorem)

Let A, B, C, A', B', C', A'', B'', C'' be the traces of the lines a, b, c, a', b', c' a'', b'' c'' respectively, then the triangles ABC and $A'B'C'$ are reciprocal *w.r.t.* the circle at infinity, hence in perspective, *i.e.*, AA', BB', CC' concur at a point O, and A'', B'', C'' are collinear.

Let the plane at infinity be taken as a plane of reference and there ABC be the triangle of reference, let the co-ordinates of O, A', B', C' refered to it be $(1, 1, 1)$, $(p, 1, 1)$, $(1, q, 1)$, $(1, 1, r)$ and then those of A'', B'', C'' are $(O, q-1, 1-r)$, $(1-p, O, r-1)$, $(1-p, q-1, O)$ respectively. Further let $(p_1, q_1, r_1, 1)$ be a point on a', $(p_2, q_2, r_2, 1)$ on b', $(p_3, q_3, r_3. 1)$ on c', then a being the secant from A to b', c' meets them in the points $(y'+p_2,\ y'q+q_2,\ y'+r_2, 1)$, $(z'+p_3,\ z'+q_3,\ z'r+r_3, 1)$ where y', z' are related as follows: $y'q+q_2 = z'+q_3$, $y'+r_2 = z'r+r_3$ giving $y' = \dfrac{r(q_2-q_3)-(r_2-r_3)}{1-qr}$. Now a'' being a secant from A'' to a, a' meets a' in the point $(p_1+xp, q_1+x, r_1+x, 1)$ where x is given by the equation,

$$\frac{q_2+y'q-q_1-x}{1-q} = \frac{r_2+y'-r_2-x}{r-1}$$

giving

$$\begin{aligned} x(2-q-r) &= y(q_2-q_3)-(r_2-r_3)-(1-q)(r_1-r_2)-(1-r)(q_1-q_2) \\ &= q_2-q_3r-(qr_2-r_3)-q_1(1-r)-r_1(1-q). \end{aligned}$$

The secant from O to a'', b'' meets them in the points $(p_1+xp,\ q_1+x+k(1-q),\ r_1+x+k(r-1), 1)$, $(p_2+y+k'(1-p),\ q_2+yq,\ r_2+y+k'(r-1), 1)$ where y has a similar value as x above and k, k' being related as follows

$$\begin{aligned} p_1+xp-k'(1-p)-p_2-y &= k(r-1)+r_1+x-k'(r-1)-r_2-y \\ &= k(1-q)+q_1+x-q_2-yq \end{aligned}$$

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eliminating k' we have

$$(1-p)\{k(2-q-r)+y(1-q)+(q_1-q_2)-(r_1-r_2)\}+(1-r)\{(k+y)(1-q)$$
$$+x(1-p)-(p_1-p_2)+(q_1-q_2)\} = 0$$

The secant from O to c'', a'' meets in the points $(p_3+z+k''(1-p),\ q_3+z+k''(q-1),\ r_3+zr,\ 1)$, $(p_1+px,\ q_1+x+h(1-q),\ r_1+x+h(r-1),\ 1)$, z having a similar value as x and h, k'' being related as k, k' above. Now these two secants from O coincide if $k=h$, *i.e.*, if $Sx(1-p)(2-q-r) \equiv S(1-p)\{(q_2-q_3)-(r_2-r_3)\}$, where S denotes the sum of similar terms.

This indentity can be easily verified with the help of the relations that x, y, z bear to p, q, r, p_i, q_i, r_i $(i=1, 2, 3)$.

This problem was given to me by Late Prof. C V. H. Rao who treated it himself synthetically (Rao, 1945, 1940) and here is the straight-forward analytic treatment.

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ON INTEGRAL REPRESENTATIONS OF THE GENERALISED k-FUNCTION OF BATEMAN AND ITS CONNECTION WITH LEGENDRE AND PARABOLIC CYLINDER FUNCTIONS

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1. Meijer (1934) has shown that the Whittaker Function

$$W_{k,m}(z) = \frac{1}{\Gamma(c)} e^{-\frac{1}{2}z} z^{k} \int_0^\infty e^{-t} t^{c-1} {}_2F_1(\tfrac{1}{2}-k+m,\ \tfrac{1}{2}-k-m\,;\ c\,,\ -t/z)\,dt,$$

$$\text{where } R(c) > 0,\ |\arg z| < \pi,\ z > 0. \qquad (1.1)$$

From the relation

$$k^{2l}_{2n}(x) = \frac{(2x)^l}{\Gamma(l+n+1)} W_{n,\,l+\frac{1}{2}}(2x), \text{ (Chakrabarty, 1952)} \qquad (1.2)$$

we derive the relation

$$k^{2l}_{2n}(x) = \frac{(2x)^{l+n} e^{-x}}{\Gamma(l+n+1)\Gamma(c)} \int_0^\infty e^{-t} t^{\alpha+m} \sum_{r=0}^{\infty} \frac{(l-n+1)_r(-l-n)_r}{r!\,(c)_r} \frac{t^{c-\alpha-m+r-1}}{(-2x)^r}\,dt \qquad (1.3')$$

$$= \frac{(-1)^m (2x)^{l+n} e^{-x}}{\Gamma(l+n+1)\Gamma(c)(1+\alpha-c)_m} \int_0^\infty e^{-t} t^{\alpha+m} \frac{d^m}{dt^m} \sum_{r=0}^{\infty} \frac{(l-n+1)_r(-l-n)_r}{r!\,(c)_r(c-\alpha)_r} \times$$

$$(c-\alpha-m)_r \frac{t^{c-\alpha+r-1}}{(-2x)^r}\,dt$$

$$= \frac{(-1)^m (2x)^{l+n} e^{-x}}{\Gamma(l+n+1)\Gamma(c)(1+\alpha-c)_m} \int_0^\infty e^{-t} t^{\alpha+m} \frac{d^m}{dt^m} \{t^{c-\alpha-1} \times$$

$${}_3F_2(l-n+1,\ -l-n,\ c-\alpha-m\,;\ c,\ c-\alpha\,;\ -\tfrac{1}{2}t/x)\}\,dt. \qquad (1.3)$$

On integration by parts m times, the above relation becomes

$$k^{2l}_{2n}(x) = \frac{(2x)^{l+n} e^{-x}}{\Gamma(l+n+1)\Gamma(c)(1+\alpha-c)_m} \int_0^\infty t^{c-\alpha-1} {}_3F_2(l-n+1,\ -l-n,\ c-\alpha-m\,;$$

$$c,\ c-\alpha\,;\ -\tfrac{1}{2}t/x) \frac{d^m}{dt^m}(e^{-t} t^{\alpha+m})\,dt$$

$$= \frac{\Gamma(m+1)(2x)^{l+n} e^{-x}}{\Gamma(l+n+1)\Gamma(c)(1+\alpha-c)_m} \int_0^\infty e^{-t} t^{c-1} L^{\alpha}_m(t)\, {}_3F_2(l-n+1,\ -l-n,\ c-\alpha-m\,;$$

$$c,\ c-\alpha\,;\ -\tfrac{1}{2}t/x)\,dt \text{ where } R(c) > R(\alpha) \geq 0$$

$$= \frac{(-1)^m \Gamma(m+\alpha+1)(2x)^{l+n} e^{-x}}{\Gamma(l+n+1)\Gamma(c)(1+\alpha-c)_m} \int_0^\infty e^{-\frac{1}{2}t} t^{c-\alpha-1} k^{\alpha-1}_{2m+\alpha+1}(\tfrac{1}{2}t) \times$$

$$ {}_3F_2(l-n+1,\ -l-n,\ c-\alpha-m;\ c,\ c-\alpha;\ -\tfrac{1}{2}t/x)\, dt$$

$$R(c) > R(\sigma) \gtrless 0. \quad (1.4)$$

Writing $n = l+1$ in (1.4) and making use of the relation

$$k^{\alpha-1}_{\alpha+1}(x) = \frac{(2x)^\alpha e^{-x}}{\Gamma(\alpha+1)},$$

we get

$$\int_0^\infty e^{-\frac{1}{2}t} t^{c-\alpha-1} k^{\alpha-1}_{2m+\alpha+1}(\tfrac{1}{2}t)\, dt = \frac{(-1)^m \Gamma(c)\Gamma(\alpha-c+m+1)}{\Gamma(m+\alpha+1)\Gamma(\alpha-c+1)};\ R(c) > R(\sigma) \geq 0. \quad (1.5)$$

Again putting $\sigma = 2l+1$, $c = l-n+1$ and $m = s-l-1$ in (1.4), we get

$$k^{2l}_{2n}(x) = \frac{(-1)^{s-l-1}\Gamma(s+l+1)(2x)^{l+n} e^{-x}}{\Gamma(l-n+1)\Gamma(s+n)} \int_0^\infty e^{-\frac{1}{2}t} t^{-l-n-1} k^{2l}_{2s}(\tfrac{1}{2}t)\left(1+\frac{t}{2x}\right)^{s+n-1} dt;$$

$$(n+l<0),\ (l-n+1>0). \quad (1\ 6)$$

Also putting $c = = l-n+1)$ and $-l-n$ in succession in (1.4), we get

$$k^{2l}_{2n}(x) = \frac{(-1)^m \Gamma(m+\sigma+1)(2x)^{l+n} e^{-x}}{\Gamma(l+n+1)\Gamma(l-n+1)(\alpha+n-l)_m} \int_0^\infty e^{-\frac{1}{2}t} t^{l-n-\alpha} k^{\alpha-1}_{2m+\alpha+1}(\tfrac{1}{2}t) \times$$

$$ {}_2F_1\left(-l-n,\ l-n-\alpha-m+1;\ l-n-\alpha+1;\ -\frac{t}{2x}\right) dt;$$

$$R(-l+n+\alpha < 1)'\ R(l-n) > -1. \quad (1.7)$$

and

$$k^{2l}_{2n}(x) = \frac{(-1)^m \Gamma(m+\alpha+1)(2x)^{l+n} e^{-x}}{\Gamma(l+n+1)\Gamma(-l-n)(l+n+\alpha+1)_m} \int_0^\infty e^{-\frac{1}{2}t} t^{-l-n-\alpha-1} k^{\alpha-1}_{2m+\alpha+1}(\tfrac{1}{2}t) \times$$

$$ {}_2F_1\left(l-n+1,\ -l-n-\sigma-m,\ -l-n-\alpha;\ -\frac{t}{2x}\right) dt$$

$$R(l+n+\alpha) < 0,\ R(l+n) < 0. \quad (1.8)$$

2. We have the relation (Whittaker and Watson, 1920)

$$k^{2l}_{2n}(\tfrac{1}{2}x) = \frac{x^l}{\Gamma(l+n+1)} W_{n,l+\frac{1}{2}}(x) = \frac{x^{l+n} e^{-\frac{1}{2}x}}{\Gamma(l+n+1)\Gamma(l-n+1)} \int_0^\infty e^{-t} t^{l-n} \left(1+\frac{t}{x}\right)^{l+n} dt;$$

$$(l-n+1>0) \quad (2.1)$$

$$= \frac{x^{l+n} e^{-\frac{1}{2}x}}{\Gamma(l+n+1)\Gamma(l-n+1)} \int_0^\infty e^{-t} \left(1+\frac{t}{x}\right)^{\alpha+m} \sum_{r=0}^{\infty} \frac{(\alpha+m-l-n)_r}{r!} \frac{t^{l-n+r}}{(-x)^r}\, dt$$

$$= \frac{x^{l+n}e^{-\frac{1}{2}x}}{\Gamma(l+n+1)\,\Gamma(l-n+m+1)}\int_0^\infty e^{-t}\left(1+\frac{t}{x}\right)^{\alpha+m}\frac{d^m}{dt^m}\sum_{r=0}^{\infty}\frac{(\alpha+m-l+n)_r}{r!}\times$$

$$\frac{(l-n+1)_r}{(l-n+m+1)_r}\,t^{l-n+m}\left(-\frac{t}{x}\right)^r dt$$

$$= \frac{x^{l+n}e^{-\frac{1}{2}x}}{\Gamma(l+n+1)\Gamma(l-n+m+1)}\int_0^\infty e^{-t}\left(1+\frac{t}{x}\right)^{\alpha+m}\times$$

$$\frac{d^m}{dt^m}\left\{t^{l-n+m}{}_2F_1\left(\alpha+m-l-n,\ l-n+1;\ l-n+m+1;\ -\frac{t}{x}\right)\right\}dt$$

$$= \frac{x^{l+n}e^{-\frac{1}{2}x}(-1)^m}{\Gamma(l+n+1)\Gamma(l-n+m+1)}\int_0^\infty t^{l-n+m}{}_2F_1\left(l-n+1,\ \alpha+m-l-n;\right.$$

$$\left.l-n+m+1;-\frac{t}{x}\right)\frac{d^m}{dt^m}\left\{e^{-t}\left(1+\frac{t}{x}\right)^{\alpha+m}\right\}dt$$

$$= \frac{(-1)^m\Gamma(m+1)\,x^{l+n-m}e^{-\frac{1}{2}x}}{\Gamma(l+n+1)\,\Gamma(l-n+m+1)}\int_0^\infty e^{-t}t^{l-n+m}\left(1+\frac{t}{x}\right)^{\alpha}L_m^{\alpha}(t+x)\times$$

$${}_2F_1\left(l-n+1,\ \alpha+m-l-n;\ l-n+m+1;\ -\frac{t}{x}\right)dt;$$

$$(l-n+m+1>0)$$

$$= \frac{\Gamma(m+\alpha+1)x^{l+n-m-\alpha}}{\Gamma(l+n+1)\Gamma(l-n+m+1)}\int_0^\infty e^{-\frac{1}{2}t}t^{l-n+m}k_{2m+\alpha+1}^{\alpha-1}\left(\frac{x+t}{2}\right)\times$$

$${}_2F_1\left(l-n+1,\ \alpha+m-l-n;\ l-n+m+1;\ -\frac{t}{x}\right)dt. \tag{2.1}$$

Putting $\alpha=2l+1$ in (2.1), we get

$$k_{2n}^{2l}(\tfrac{1}{2}x) = \frac{\Gamma(m+2l+2)x^{n-m-l-1}}{\Gamma(l+n+1)\Gamma(l-n+m+1)}\int_0^\infty e^{-\frac{1}{2}t}t^{l-n+m}k_{2m+2l+2}^{2l}\left(\frac{x+t}{2}\right)\left(1+\frac{t}{x}\right)^{n-l-1}dt. \tag{2 2}$$

Writing $n=l+1$ in (2.1), we get

$$\int_0^\infty e^{-\frac{1}{2}t}t^{m-1}k_{2m+\alpha+1}^{\alpha-1}\left(\frac{x+t}{2}\right)dt = \frac{\Gamma(m)}{\Gamma(m+\alpha+1)}x^{m+\alpha}e^{-\frac{1}{2}x};\quad (m>0,\ m+\alpha>0. \tag{2 3}$$

Again putting $\alpha=l+n-m$ in (2.1), we obtain

$$k_{2n}^{2l}(\tfrac{1}{2}x) = \frac{1}{\Gamma(l-n+m+1)}\int_0^\infty e^{-\frac{1}{2}t}t^{l-n+m}k_{l+n+m+1}^{l+n-m-1}\left(\frac{x+t}{2}\right)dt.$$

If we put $m=0$ in (2.3) and use the relation (1.2), we derive the relation (1.3′).

The relation (2.3) can be put in the form

$$\frac{\Gamma(m)2^{\alpha}x^{m+\alpha}e^{-x}}{\Gamma(m+\alpha+1)} = \int_0^{\infty} e^{-t}t^{m-1}k_{2m+\alpha+1}^{\alpha-1}(x+t)\,dt; \quad (m>0,\ m+\alpha>0). \tag{2.4}$$

Differentiating both sides m times, which is permissible, we get

$$\frac{\Gamma(m)\Gamma(m+1)}{\Gamma(m+\alpha+1)}(2x)^{\alpha}e^{-x}L_m^{\alpha}(x) = \int_0^{\infty} e^{-t}t^{m-1}\frac{d^m}{dx^m}k_{2m+\alpha+1}^{\alpha-1}(x+t)\,dt \quad m>0,\ \alpha>0). \tag{2.5}$$

Writting n for $m+\alpha$ and x^2-1 for x in (2.4) and then differentiating both sides n times, we obtain

$$\frac{\Gamma(m)}{\Gamma(n+1)}2^{n-m}\frac{d^n}{dx^n}(x^2-1)^n = \int_0^{\infty} e^{-(t+1)}t^{m-1}\frac{d^n}{dx^n}\left\{e^{x^2}k_{n+m+1}^{n-m-1}(t+x^2-1)\right\}dt. \tag{2.6}$$

Since $P_n(x) = \dfrac{1}{2^n\Gamma(n+1)}\dfrac{d^n}{dx^n}(x^2-1)^n$, we get

$$\Gamma(m)2^{2n-m}P_n(x) = \int_0^{\infty} e^{-(t+1)}t^{m-1}\frac{d^n}{dx^n}\left\{e^{x^2}k_{n+m+1}^{n-m-1}(t+x^2-1)\right\}dt, \quad (m>0). \tag{2.7}$$

3. Let us now use the relation (Dhar, 1938)

$$x^{\lambda-1}e^{\frac{1}{2}x}M_{k,m}(x) \doteqdot \Gamma(\lambda+m+\tfrac{1}{2})(p-1)^{k-\lambda-m-k+\frac{1}{2}}p\,{}_2F_1(m-\lambda+\tfrac{1}{2},\ k+m+\tfrac{1}{2};\,2m+1;\,1/p) \tag{3.1}$$

and

$$k_{2n}^{2l}(\tfrac{1}{2}x) = (-1)^{n-l-1}x^lM_{n,l+\frac{1}{2}}(x)/\Gamma(2l+2), \tag{3.2}$$

we derive the relation

$$x^{\lambda-l-1}k_{2n}^{2l}(x) \doteqdot \frac{(-1)^{n-l-1}2^{2l+1}\Gamma(l+\lambda+1)}{\Gamma(2i+2)}\,\frac{p(p-1)^{n-\lambda}}{(p+1)^{l+n+1}} \times {}_2F_1(l-\lambda+1,\ l+n+1;\,2l+2;\,2/(p+1)). \tag{3.3}$$

Putting $n=0$ and $\lambda=-m$ in the above relation, we get

$$x^{-m-l-1}k_0^{2l}(x) \doteqdot \frac{(-1)^{l+1}2^{2l+1}\Gamma(l-m+1)}{\Gamma(2l+2)}\,\frac{p(p-1)^m}{(p+1)^{l+1}} \times {}_2F_1(l+m+1,\ l+1;\,2l+2;\,2/p+1) \tag{3.4}$$

Now using the relation [MacRobert, 1938]

$$Q_n^m(x) = \frac{\Gamma(m+n+1)\Gamma(\frac{1}{2})(x^2-1)^{\frac{1}{2}m}}{2^{n+1}\Gamma(n+\frac{3}{2})(x+1)^{m+n+1}}\,{}_2F_1(m+n+1,\ n+1;\,2n+2;\,2/(x+1)), \quad |x+1|>2$$

we get

$$x^{-l-m-1}k_0^{2l}(x) \doteqdot \frac{(-1)^{l+1}2^{3l+2}}{\pi^{\frac{1}{2}}}\,\frac{\Gamma(l-m+1)\Gamma(l+\frac{3}{2})}{\Gamma(2l+2)\Gamma(l+m+1)}\,p(p^2-1)^{\frac{1}{2}m}Q_l^m(p), \quad |p+1|>2. \tag{3.5}$$

4. We know that if

$$h(x) \doteqdot f(p),$$

then

$$\tfrac{1}{2}\pi^{\frac{1}{2}}p^{\frac{1}{2}m}f(p^{\frac{1}{2}}) \doteqdot (2x^{\frac{1}{2}})^{-m-1}\int_0^\infty h(s)e^{-\frac{1}{4}s^2/x}H_m\left(\frac{s}{2x^{\frac{1}{2}}}\right)ds, \tag{4.1}$$

where $H_m(x)$ is the Hermite polynomial defined by

$$H_m(x) = (-1)^m e^{x^2}\frac{d^m}{dx^m}(e^{-x^2}). \tag{4.2}$$

Let

$$h(s) = e^{-\frac{1}{2}s}k_{2n}^{2l}(\tfrac{1}{2}s) \doteqdot \frac{(-1)^{n-l-1}p^{n-l}}{(p+1)^{n+l+1}} = f(p).$$

Hence from (4.1) we get

$$(2x^{\frac{1}{2}})^{-(m+1)}\int_0^\infty e^{-\frac{1}{2}(s+\frac{1}{2}s^2/x)}k_{2n}^{2l}(\tfrac{1}{2}s)H_m\left(\frac{s}{2x^{\frac{1}{2}}}\right)ds$$

$$\doteqdot (-1)^{n-l-1}\tfrac{1}{2}\pi^{\frac{1}{2}}\frac{p^{\frac{1}{2}(m+n-l)}}{(1+p^{\frac{1}{2}})^{n+l+1}}$$

$$\doteqdot (-1)^{n-l-1}\tfrac{1}{2}\pi^{\frac{1}{2}}\frac{p^{\frac{1}{2}}}{(1+p^{\frac{1}{2}})^{2l+2-m}}\left(1-\frac{1}{1+p^{\frac{1}{2}}}\right)^{m+n-l-1}$$

$$\doteqdot (-1)^{n-l-1}\tfrac{1}{2}\pi^{\frac{1}{2}}\sum_{r=0}^{m+n-l-1}(-1)^r\binom{m+n-l-1}{r}\frac{p^{\frac{1}{2}}}{(1+p^{\frac{1}{2}})^{2l-m+r+2}}. \tag{4.2}$$

On interpetation by the relation (Mclachlan and Humbert, 1941, 1950)

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{p^{\frac{1}{2}}}{(1+p^{\frac{1}{2}})^n} \doteqdot (2x)^{\frac{1}{2}(n-1)}e^{\frac{1}{2}x}D_{-n}((2x)^{\frac{1}{2}}), \tag{4.3}$$

we derive the relation

$$\int_0^\infty e^{-\frac{1}{2}(s+\frac{1}{2}s^2/x)}k_{2n}^{2l}(\tfrac{1}{2}s)H_m(\tfrac{1}{2}s/x^{\frac{1}{2}})\,ds$$

$$= 2^{\frac{1}{2}m}(2x)^{l+1}e^{\frac{1}{2}x}\sum_{r=0}^{m+n-l-1}(-1)^{n-l+r-1}\binom{m+n-l-1}{r}(2x)^{\frac{1}{2}r}D_{-(2l-m+r+2)}((2x)^{\frac{1}{2}}). \tag{4.4}$$

Putting $m = 0$, in the above, we obtain

$$\int_0^\infty e^{-\frac{1}{2}(s+\frac{1}{2}s^2/x)}k_{2n}^{2l}(\tfrac{1}{2}s)\,ds = (2x)^{l+1}e^{\frac{1}{2}x}\sum_{r=0}^{n-l-1}(-1)^{n-l+r-1}\binom{n-l-1}{r}D_{-(2l+r+2)}((2x)^{\frac{1}{2}}) \tag{4.5}$$

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SOME SELF-RECIPROCAL FUNCTIONS.

By

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1. Introduction. We have proved before (Bhatnagar, 1954) that the kernels $\varpi_{\mu,\nu}(x)$, $\varpi_{\mu,\nu,\lambda}(x)$ and $\varpi_{n_1,n_2 \ldots n_n}(x)$ defined as below play the role of transforms:

$$(i) \quad \varpi_{\mu,\nu}(x) = x^{\frac{1}{2}} \int_0^\infty J_\nu(t) J_\mu\left(\frac{x}{t}\right)\frac{dt}{t} \qquad (\mu, \nu > -\tfrac{1}{2})$$

$$(ii) \quad \varpi_{\mu,\nu,\lambda}(x) = \int_0^\alpha \frac{J_\lambda(t)}{t^{\frac{1}{2}}} \varpi_{\mu,\nu}\left(\frac{x}{t}\right) dt \qquad (\mu, \nu, \lambda > -\tfrac{1}{2})$$

$$(iii) \quad \varpi_{n_1,n_2 \ldots n_n}(x) = \int_0^\infty \frac{J_{n_n}(t)}{t^{\frac{1}{2}}} \varpi_{n_1,n_2 \ldots n_{n-1}}\left(\frac{x}{t}\right) dt \qquad (n_1, n_2 \ldots n_n > -\tfrac{1}{2}).$$

If a function satisfies the equation

$$f(x) = \int_0^\infty f(x)\, \varpi_{\mu,\nu}(xt)\, dt$$

we say that it is $R_{\mu,\nu}$ *i.e.* self-reciprocal under $\varpi_{\mu,\nu}$ transform. Similarly if a function is, self-reciprocal in $\varpi_{\mu,\nu,\lambda}$ transform or $\varpi_{n_1,n_2,\ldots n_n}$ transform, we say it is $R_{\mu,\nu,\lambda}$ or $R_{n_1,n_2,\ldots n_n}$. In the previons papers (Bhatnagar) we have given a number of examples of functions self-reciprocal in $\varpi_{\mu,\nu}$ or $\varpi_{\mu,\nu,\lambda,\zeta}$ transforms.

The object of this paper is, firstly, to obtain relations between these kernels and $J_\nu(x)$ and secondly by the help of these relations to find out some new functions self-reciprocal in the Hankel or in these transforms. It is interesting to note that some of the known functions self-reciprocal in the Hankel transform come out as particular cases of our more general examples. Many of the functions obtained here are in the form of integrals and it may be noticed that it is not easy to evaluate them.

2. Relations between $J_\nu(x)$ and other kernels.

$$(i) \quad \varpi_{\nu-1,\nu}(x) = J_{2\nu-1}(2x^{\frac{1}{2}}); \qquad (\nu \geq \tfrac{1}{2})$$

$$(ii) \quad \varpi_{\nu-4/3,\, \nu-2/3,\, \nu}(x) = (x)^{-1/6} J_{3\nu-2}(3x^{\frac{1}{3}}); \qquad (\nu \geq \tfrac{5}{6})$$

$$(iii) \quad \varpi_{\nu-3/2,\, \nu-1,\, \nu-\frac{1}{2},\, \nu}(x) = (x)^{-\frac{1}{4}} J_{4\nu-3}(4x^{\frac{1}{4}}); \qquad (\nu \geq 1)$$

$$(iv) \quad \varpi_{\nu-(2(n-1)/n),\ldots,\nu-2/n,\, \nu}(x) = (x)^{-((n-2)/2n)} J_{n\nu-(n-1)}(nx^{1/n}); \; (\nu \geq 3n-4/2n)$$

$$(n=2, 3, 4\ldots n)$$

(v) $\varpi_{\nu-1,\nu,-1,\zeta}(x) = (x)^{-\frac{1}{4}}\varpi_{2\nu-1,2\zeta-1}(4x^{\frac{1}{4}})$; $\begin{bmatrix}\nu\geq\frac{1}{2}\\ \zeta\geq\frac{1}{2}\end{bmatrix}$

(vi) $\varpi_{\mu-1\,\mu,\nu-1,\nu,\lambda-1,\lambda}(x) = 2^{\frac{1}{4}}(x)^{-\frac{1}{4}}\varpi_{2\mu-1,2\nu-1,2\lambda-1}(2^3x^{\frac{1}{4}};)$ $\begin{bmatrix}\nu\geq\frac{1}{2},\ \lambda\geq\frac{1}{2}\\ \mu\geq\frac{1}{2}\end{bmatrix}$

(vii) $\varpi_{\mu_1-1,\mu_1,\mu_2-1.\mu_2\ .\mu_{2n}-1,\mu_{2n}}(x) = 2^{n/2-1}(x)^{-1/4}\varpi_{2\mu_1-1,2\mu_2-1,\,.\,2\mu_n-1}(2^n x^{\frac{1}{4}})$; $(n=2,3,4\ldots n)$.

3. **A list of new self-reciprocal functions.**

(1) We have proved in the paper (Bhatnagar) that

$$g(x) = x^{n+\mu+\frac{1}{2}}\int_0^\infty y^{m+\lambda-n-\mu-1}K_m(y)K_n\left(\frac{x}{y}\right); \qquad \begin{bmatrix}\nu=\mu+2n,\\ \zeta=\lambda+2m\end{bmatrix}$$

is $R_{\mu,\nu,\lambda,\zeta}$.

As a particular case, if

(i) $\mu=\nu-1$, $\lambda=\zeta-1$ we prove that (by relation (v) above)

$$x^{\nu+\zeta-\frac{1}{2}}K_{\zeta-\nu}(x); \qquad (\nu\geq\tfrac{1}{2},\ \zeta\geq\tfrac{1}{2})$$

is $R_{2\nu-1,2\zeta-1}$.

(ii) further, if $\nu=\zeta-\frac{1}{2}$ we prove the well-known result

$$x^{4\zeta-5/2}\,e^{-\frac{1}{2}x^2}; \qquad (\zeta\geq 1)$$

is $R_{4\zeta-3}$

(2) It can be easily proved that (Bhatnagar)

(i) $$g(x) = x^{n+\lambda+\frac{1}{2}}\int_0^\infty y^{-n-\lambda-3/2+\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}}J_{\frac{1}{2}(\mu+\nu)}(y)\,K_n\left(\frac{x}{y}\right)dy; \quad \zeta=\lambda+2n;\ n\geq 0$$

$$(\lambda+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu+3/2>0)$$

and (ii) $$g(x)=x^{n+\lambda+\frac{1}{2}}\int_0^\infty y^{-n-\lambda-3/2+\nu/2-\mu/2+\frac{1}{2}}J_{\frac{1}{2}(\mu+\nu)}(y)\,K_n\left(\frac{x}{y}\right)dy; \quad (\zeta=\lambda+2n;\ n\geq 0;$$

$$\lambda+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+3/2>0).$$

are $R_{\mu,\nu,\lambda,\zeta}$

As particuar cases of these examples we obtain (by relation (i) and (v)) new functions self-reciprocal in the Hankel transform.

(iii) If in example (i) above we take $\mu=\nu-1$, $\lambda=\zeta-1$ we easily prove that

$$x^{2\zeta-\frac{1}{2}}\int_0^\infty y^{-\frac{1}{2}-\zeta}J_{\nu-\frac{1}{2}}(y)\ e^{-(x^2/(4y))}$$

is $R_{2\nu-1,2\zeta-1}$.

(iv) Further if, $\zeta=\nu-\frac{1}{2}$, we prove that

$$x^{4\nu-5/2}\int_0^\infty y^{-\nu}J_{\nu-\frac{1}{2}}(x)\,e^{-x^4/(16y)}\ dy; \qquad (\nu\geq 5/8)$$

is $R_{4\nu-3}$,

or, if $\nu=\zeta-\frac{1}{2}$ we prove that

$$x^{4\zeta-\frac{1}{2}}\int_0^\infty y^{-\frac{1}{2}-\zeta}\,J_{\zeta-1}(x)\,e^{-x^2/(16y)}\,dy\,; \qquad (\zeta\geq 5/8)$$

is $R_{4\zeta-3}$.

In particular, if $\nu=1$ in the first, we prove that

$$x^{-\frac{1}{2}}\exp\{-x^2/(2\sqrt{2})\}\sin(x^2/2\sqrt{2})^*$$

is R_1.

(*v*) If in example (*ii*) above, we take $\mu=\nu-1$, $\lambda=\zeta-1$, we easily prove that

$$x^{2\zeta-\frac{1}{2}}\int_0^\infty y^{-\zeta+\frac{1}{2}}\,J_{\nu-\frac{1}{2}}(y)\,e^{-x^2/(4y)}\,dy$$

is $R_{2\nu-1,\,2\zeta-1}$.

In particular, if $\nu=\zeta-\frac{1}{2}$, we prove that

$$x^{2\zeta-\frac{1}{2}}\int_0^\infty y^{-\zeta+\frac{1}{2}}\,J_{\zeta-1}(y)\,e^{-x^2/(4y)}\,dy$$

is $R_{2\zeta-2,\,2\zeta-1}$. For $\zeta=3/2$, we obtain

$$x^{7/2}\exp\{-x^2/(2\sqrt{2})\}\sin\{x^2/(2\sqrt{2})\}$$

which is R_8

(*vi*) If, $\zeta=\nu-\frac{1}{2}$ in the previous example, we prove that

$$x^{2\nu-3/2}\int_0^\infty y^{-\nu+1}\,J_{\nu-\frac{1}{2}}(x)\,e^{-x^2/(4y)}\,dy$$

is $R_{2\nu-1,\,2\nu-2}$

For, $\nu=2$ we get the result obtained by Gupta (1942) that is

$$x^{11/2}\,J_{3/2}\left(\frac{x^2}{2\sqrt{2}}\right)\,K_{3/2}\left(\frac{x^2}{2\sqrt{2}}\right)$$

is R_5

(3) It can be proved (Bhatnagar) easily that

$$g(x)=x^{\frac{1}{2}(\mu-\nu+1)}\int_0^\infty y^{\frac{1}{2}(\lambda-\zeta-\mu+\nu-2)}\,J_{\frac{1}{2}(\mu+\nu)}\left(\frac{x}{y}\right)J_{\frac{1}{2}(\lambda+\zeta)}(y)\,dy, \qquad \begin{bmatrix}\lambda+\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}>0\\ \mu+\frac{1}{2}\zeta-\frac{1}{2}\lambda+3/9>0\end{bmatrix}$$

is $R_{\mu,\,\nu,\lambda,\,\zeta}$

(*i*) In particular, if $\mu=\nu-1$, $\lambda=\zeta-1$, we prove that

$$x^{-\frac{1}{2}}\,\varpi_{\nu-\frac{1}{2},\,\zeta-\frac{1}{2}}\left(\frac{x^2}{4}\right)$$

is $R_{2\nu-1,\,2\zeta-1}$.

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* These results have been obtained by Gupta (1946) also but in a different manner.

Further, if $\nu=\zeta-\frac{1}{2}$ we prove that

$$x^{-\frac{1}{2}}\,\varpi_{\zeta-\frac{1}{2},\,\zeta-1}\left(\frac{x^4}{16}\right)$$

is $R_{4\zeta-3}$

(4) Taking $g(x)$ as in eaxmple (1) and putting $\lambda=\zeta-\frac{1}{2}$, $\nu=\zeta-1$, $\mu=\zeta-3/2$, and applying the relations (iii) of Art 2, we prove that

$$x^{4\zeta-3/2}\int_0^\infty K_{\frac{1}{4}}(y)\,K_{\frac{1}{4}}\left(\frac{x^4}{16y}\right)dy\,,\qquad(\zeta\geq 1)$$

is $R_{4\zeta-3}$.

(5) Taking $g(x)$ as in example (2) and putting $\lambda=\zeta-\frac{1}{2}$, $\nu=\zeta-1$, $\mu=\zeta-3/2$ and applying the relations (iii) of Art 2, we prove that

$$x^{4\zeta+5/2}\int_0^\infty y^{-\zeta-1}\,J_{\zeta-5/4}(x)\,K_{\frac{1}{4}}\left(\frac{x^4}{16y}\right)dy$$

is $R_{4\zeta-3}$. For, $\lambda=\zeta-1$, $\nu=\zeta-\frac{1}{2}$, $\mu=\zeta-3/2$ (Bhatnagar) we prove that

$$x^{4\zeta-\frac{1}{2}}\int_0^\infty y^{-\frac{1}{2}-\zeta}\,J_{\zeta-1}(y)\,e^{-x^4/(16y)}dy$$

is $R_{4\zeta-3}$.

(6) (i) If we take $g(x)$ as defined in example (3) and put $\lambda=\zeta-\frac{1}{2}$, $\nu=\zeta-1$, $\mu=\zeta-3/2$ then by relation (iii) of Art 2, we prove that

$$x^{5/2}\int_0^\infty J_{\zeta-\frac{1}{4}}(x)\,J_{\zeta-5/4}\left(\frac{x^4}{16y}\right)\frac{dy}{y}\;;\qquad(\zeta\geq 1)$$

is $R_{4\zeta-3}$. This can be put (by relation (i)) into the known form

$$x^{\frac{1}{2}}\,J_{2\zeta-3/2}\left(\tfrac{1}{2}x^2\right)$$

which is $R_{4\zeta-3}$.

(ii) If $\lambda=\zeta-1$, $\nu=\zeta-\frac{1}{2}$, $\mu=\zeta-3/2$ we easily see that

$$x^{4\zeta-5/2}\int_0^\infty y^{-\frac{1}{2}}\,e^{-y-x^4/16y}\,dy$$

is $R_{3\zeta-3}$. This integral on evaluation gives the known result, i.e.,

$$x^{\nu+\frac{1}{2}}\,e^{-\frac{1}{2}x^2}$$

is R_ν.

(7) We have proved (Bhatnagar) that

$$g(x)=x^{\nu+\mu+\frac{1}{2}}\int_0^\infty y^{\lambda-\mu-n-1}K_n\left(\frac{x}{y}\right)e^{-\frac{1}{2}y^2}dy\qquad\begin{bmatrix}\nu=\mu+2n\\ n\geq 0\end{bmatrix}$$

is $R_{\mu,\,\nu,\,\lambda}$.

In particular, if $\nu=\lambda-2/3, \mu=\lambda-4/3$ then (by relation (*ii*) of Art 2) we prove that

$$x^{3\lambda-\frac{1}{2}}\int_0^\infty K_{\frac{1}{3}}\left(\frac{x^3}{3\sqrt{3y}}\right)e^{-\frac{1}{4}y^2}dy\,; \qquad \lambda \geq 5/6$$

is $R_{3\lambda-2}$.

(8) We have proved (Bhatnagar) that

$$g(x)= x^{\frac{1}{2}(\mu-\nu+1)}\int_0^\infty y^{\lambda-\mu/2+\nu/2-1}\, e^{-\frac{1}{4}y^2}J_{\frac{1}{2}(\mu+\nu)}\left(\frac{x}{y}\right)dy$$

is $R_{\mu,\,\nu,\,\lambda}$.

In particular, if $\nu=\lambda-2/3$, $\mu=\lambda-4/3$ then applying the relation (*ii*) of Art 2, we prove that

$$x^{3/2}\int_0^\infty y^{\lambda-\frac{3}{2}}\, e^{-\frac{1}{4}y^2}J_{\lambda-1}\left(\frac{x^3}{3\sqrt{3y}}\right)dy$$

is $R_{3\lambda-2}$

(9) We have proved (Bhatnagar) that

$$g(x)=x^{\mu+\frac{1}{2}}\int_0^\infty y^{-\mu-1}\, J_{\frac{1}{2}\nu}(\tfrac{1}{2}y^2)e^{-x^2/(2y^2)}\; dy$$

is $R_{\mu,\,\nu}$. In particular, if $\mu=0$, we easily prove that.

$$x^{\frac{1}{2}}\, J_{\frac{1}{2}\nu}\left(\frac{x}{\sqrt{2}}\right)K_{\frac{1}{2}\nu}\left(\frac{x}{\sqrt{2}}\right)$$

is $R_{0,\,\nu}$.

(10) We know that (Bhatnagar)

$$x^{-\frac{1}{2}\nu}H_{\frac{1}{2}(\nu-1)}(x)$$

is $R_{1,\,\nu}$. In particular, if $\nu=2$

$$x^{-3/2}H_{\frac{1}{2}}\,(\tfrac{1}{2}x^2)\equiv x^{-5/2}\,\sin^2\,(\tfrac{1}{4}x^2)$$

is R_{3}. and if $\nu=0$.

$$x^{\frac{1}{2}}\,H_{-\frac{1}{2}}(\tfrac{1}{2}x^2) \quad \text{(Gupta, 1946)}$$

is R_1.

(11) We also know that (Bhatnagar)

$$x^{\frac{1}{2}(\mu-\nu+1)}\,J_{\frac{1}{2}(\mu+\nu)}\,(x)$$

is $R_{\mu,\nu}$. In particular, $\mu=\nu-1$,

$$x^{\frac{1}{2}}\,J_{\nu-\frac{1}{2}}\,(\tfrac{1}{2}x^2)$$

is $R_{2\nu-1}$.

(12) We have proved (Bhatnagar) that

$$x^{\mu+\frac{1}{2}}\int_0^\infty y^{-\mu-2}\,D^2{}_{2m+1}(x)\,e-\tfrac{1}{2}x^2/y^2\,dy$$

$R_{\mu,1}$. In particular, if $\mu=0$

$$x^{8/2}\int_0^\infty y^{-2}\,D^2{}_{2m+1}(y)e-x^4/8y^2dy$$

is R_1, and if $\mu=2$,

$$x^{11/2}\int_0^\infty y^{-4}\,D^2{}_{2m+1}(y)\,e^{-x^4/(8y)}\,dy$$

is R_8.

4. Some functions self-reciprocal under $\varpi_{\mu,\nu,\lambda}$ transform

(i) $$\int_0^\infty \frac{J_{\frac{1}{2}\nu}(\frac{1}{2}t^2)}{t^{\frac{1}{2}}}\left[I_0\left(\frac{y}{t}\right)-L_0\left(\frac{y}{t}\right)\right]dt\,; \qquad R_{\nu,\frac{1}{2},\frac{1}{2}}$$

(ii) $$\int_0^\infty \frac{J_{\frac{1}{2}\nu}(\frac{1}{2}t^2)}{t^{\frac{1}{2}}}\left(\frac{y}{t}\right)^{\frac{1}{2}}\left[I_{-\frac{1}{2}}\left(\frac{1}{2}\frac{y}{t}\right)K_{-\frac{1}{2}}\left(\frac{1}{2}\frac{y}{t}\right)\right]dt; \qquad R_{-\frac{1}{2},-\frac{1}{2},\nu}$$

(iii) $$\int_0^\infty \frac{J_{\frac{1}{2}\nu}(\frac{1}{2}t^2)}{t^{\frac{1}{2}}}\left\{J_0{}^2\left(\frac{1}{2}\frac{y}{t}\right)+y_0^2\left(\frac{1}{2}\frac{y}{t}\right)\right\}dt\,; \qquad R_{-\frac{1}{2},-\frac{1}{2},\nu}$$

(iv) $$\int_0^\infty \frac{J_{\frac{1}{2}\nu}(\frac{1}{2}t^2)}{t^{\frac{1}{2}}}\left(\frac{y}{t}\right)^{-\frac{1}{2}}H_0\left(\frac{y}{t}\right)dt\,; \qquad R_{1,1,\nu}$$

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A GENERAL THEOREM

By

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Introduction. We here prove a general theorem from which some theorems proved by us (Bhatnagar, 1953) in a different manner can be deduced at once.

Theorem. *Let*

$$\psi(p) = \int_0^\infty \frac{t^{\nu+3/2} f(t)\, dt}{(t^2+p^2)^{\nu+3/2}}.$$

If $f(t)$ *is* $R_{n_1, n_2, \ldots n_{n-1}, \nu+\frac12}$, *then* $p^{\nu+\frac12}\psi(p)$ *is* $R_{n_1, n_2, \ldots n_{n-1}, \nu}$.

We have

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac14 ns}\, \Gamma(\tfrac12 n_1 + \tfrac12 s + \tfrac14) \ldots \Gamma(\tfrac12 n_{n-1} + \tfrac12 s + \tfrac14)\, \Gamma(\tfrac12 \nu + \tfrac12 + \tfrac12 s + \tfrac14)\, \psi(s)\, t^{-s}\, ds.$$

Hence

$$\psi(p) = \int_0^\infty \frac{t^{\nu+3/2}\, dt}{(t^2+p^2)^{\nu+3/2}} \, \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac14 ns}\, \Gamma(\tfrac12 n_1 + \tfrac12 s + \tfrac14) \ldots \Gamma(\tfrac12 n_{n-1} + \tfrac12 s + \tfrac14) \times$$

$$\times \Gamma(\tfrac12 \nu + \tfrac12 + \tfrac12 s + \tfrac14)\, \psi(s)\, t^{-s}\, ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac14 ns}\, \Gamma(\tfrac12 n_1 + \tfrac12 s + \tfrac14) \ldots \Gamma(\tfrac12 n_{n-1} + \tfrac12 s + \tfrac14) \times$$

$$\times \Gamma(\tfrac12 \nu + \tfrac12 + \tfrac12 s + \tfrac14)\, \psi(s)\, ds \int_0^\infty \frac{t^{\nu+3/2-s}\, dt}{(t^2+p^2)^{\nu+3/2}}.$$

Therefore

$$y^{\nu+\frac12}\psi(p) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac14 ns}\, \Gamma(\tfrac12 n_1 + \tfrac12 s + \tfrac14) \ldots \Gamma(\tfrac12 n_{n-1} + \tfrac12 s + \tfrac14)\, \Gamma(\tfrac12 \nu + \tfrac12 s + \tfrac14)\, p^{-s}\, \psi_1(s)\, ds$$

where $\psi_1(s) = \psi(s)\, \dfrac{\Gamma(\frac12\nu + \frac12 s + \frac34)\ \Gamma(\frac12\nu - \frac12 s + \frac54)}{2\Gamma(\nu+\frac32)}$,

that is, $\psi_1(s) = \psi_1(1-s)$

so that $p^{\nu+\frac12}\psi(p)$ is $R_{n_1, n_2, \ldots n_{n-1}, \nu}$.

The change in the order of integration can be easily justified if $f(t)$ belongs to $A(a, u)$ (Titchmarsh, 1948) and $\psi(t)$ is $O[e^{(\frac14 n\pi - \omega + \eta)|t|}]$ and $\nu + \frac54 - c > 0$, $\nu + c > -\frac12$.

Deductions.

(i) *If* $f(t)$ *is* $R_{\nu+1}$, *then* $p^{\nu+\frac{1}{2}}\psi(p)$ *is* R_{ν}.

(ii) *If* $f(t)$ *is* $R_{\nu-1,\nu+1}$, *then* $p^{2\nu+3/2}\psi(\frac{1}{2}p^2)$ *is* $R_{2\nu-1}$.

(iii) *If* $f(t)$ *is* $R_{\nu+1,\nu+1}$, *then* $p^{2\nu+3/2}\psi(\frac{1}{2}p^2)$ *is* $R_{2\nu+1}$.

(iv) *If* $f(t)$ *is* $R_{\nu-1/3,\nu-\frac{1}{3},\nu+1}$, *then* $p^{3\nu+5/2}\psi(p^3/3\sqrt{3})$ *is* $R_{3\nu-2}$.

(v) *If* $f(t)$ *is* $R_{\nu-2(n-1/n),\nu-2/n,\nu+1}$; *then* $p^{n(\nu+1)-\frac{1}{2}}\psi(p^n/n^{n/2})$ *is* $R_{n\nu-(n-1)}$, $(n = 2, 3\ldots)$,

Examples.

(i) $f(t) = e^{-\frac{1}{4}t^2}t^{\nu+3/2}$; $R_{\nu+1}$ *then*, $p^{\nu+\frac{1}{2}}\psi(p) = p^{\nu+\frac{1}{2}}e^{\frac{1}{4}p^2}D_{-2\nu-3}(p)$ *is* R_{ν}.

(ii) $*f(t) = t^{-\frac{1}{2}}J_{\nu}(t)$; $R_{\nu-1,\nu+1}$ *then*, $\psi(p) = \int_0^\infty \frac{t^{\nu+1}J_{\nu}(t)\,dt}{(t^2+p^2)^{\nu+3/2}}$

i.e., $p^{2\nu+3/2}\psi(\frac{1}{2}p^2)$ is $R_{2\nu-1}$.

(iii) $*f(t) = t^{\frac{1}{2}}J_{\nu+1}(t)$; $R_{\nu+1,\nu+1}$, then $\psi(p) = \int_0^\infty \frac{t^{\nu+2}J_{\nu+1}(t)\,dt}{(t^2+p^2)^{\nu+3/2}}$

i.e., $p^{2\nu+3/2}\psi(\frac{1}{2}p^2)$ is $R_{2\nu+1}$.

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DEPARTMENT OF MATHEMATICS & STATICS,
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* In view of the fact that $f(t)$'s are different, $\psi(p)$'s are also different.

SOME PROPERTIES OF FIBONACCI NUMBERS—I

By

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In a recent paper the author (Subba Rao, 1953) derived among other results, the summations of several series involving Fibonacci numbers. In this paper, the author studies some properties of Fibonacci numbers which are of a general nature.

Proposition 1. *Between n and 2n, there lie at least one and at the most two Fibonacci numbers, n being an integer* > 1.

Proof. The proof is divided into three parts.

Case I. Let $u_{m+1} > \frac{1}{2}(u_{m+1} + u_m) = \frac{1}{2} u_{m+2} > n \geq u_m$.

By using the recurring relation $u_{m+1} = u_m + u_{m-1}$, we see that only one Fibonacci number, u_{m+1}, lies between n an $2n$, in this case.

Case II. Let $u_{m+1} > n > \frac{1}{2}(u_{m+1} + u_m) = \frac{1}{2} u_{m+2} > u_m$.

In this case, the two Fibonacci numbers, u_{m+1} and u_{m+2}, lie between n and $2n$.

Case III. Let $2n$ be a Fibonacci number $= u_m$, say.

In this case, only one Fibonacci number, u_{m-1}, lies between n and $2n$.

Proposition 2. *The number of Fibonacci numbers not exceeding n is asymptotic to log n/log a, where* $a = \frac{1}{2}(1 + \sqrt{5})$.

Proof. Let
$$u_m \leq n < u_{m+1}.$$

It is known (Subba Rao, 1953) that

$$u_m = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{m+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{m+1} \right\}.$$

Therefore u_m will be $\leq n$

if
$$\left(\frac{1+\sqrt{5}}{2} \right)^{m+1} \leq \sqrt{5}\, n + \left(\frac{1-\sqrt{5}}{2} \right)^{m+1}$$

$$< (\sqrt{5} + \varepsilon_1)\, n, \quad \text{where } 0 < \varepsilon_1 < 1.$$

i.e., if
$$m < \frac{\log n}{\log a} + \frac{\log(\sqrt{5} + \varepsilon_1)}{\log a} - 1 \tag{1}$$

And u_{m+1} will be $> n$ if

$$\left(\frac{1+\sqrt{5}}{2}\right)^{m+2} > \sqrt{5}n + \left(\frac{1-\sqrt{5}}{2}\right)^{m+2}$$

$$> (\sqrt{5}+\epsilon_2)n,$$

where $\epsilon_2 = \left(\frac{1-\sqrt{5}}{2}\right)\epsilon_1$ and $0 < |\epsilon_2| < 1$;

i.e., if

$$m > \frac{\log n}{\log a} + \frac{\log(\sqrt{5}+\epsilon_2)}{\log a} - 2 \qquad (2)$$

From (1) and (2) it follows that m, the number of Fibonacci numbers not exceeding n, is asymptotic to $\frac{\log n}{\log a}$, where $a = \frac{1}{2}(1+\sqrt{5})$.

Proposition 3. *Given a Fibonacci number, to find its rank.*

Proof. Let the given Fibonacci number be k and its rank be n.

Then

$$k = u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\}$$

$$\therefore \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} = \sqrt{5}k + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} < (\sqrt{5}+\tfrac{1}{2})k$$

$$\therefore \quad n < \frac{\log k}{\log a} + \frac{\log(\sqrt{5}+\frac{1}{2})}{\log a} - 1 \qquad (1)$$

where $a = \frac{1}{2}(1+\sqrt{5})$

Also

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} = \sqrt{5}k + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} > (\sqrt{5}-\tfrac{1}{2})\,k.$$

$$\therefore \quad n > \frac{\log k}{\log a} + \frac{\log(\sqrt{5}-\frac{1}{2})}{\log a} - 1. \qquad (2)$$

It is obvious that between the two expressions on the right hand side of (1) and (2), only one integer can lie. Since n is an integer, it is therefore the greatest integer in

$$\frac{\log k}{\log a} + \frac{\log(\sqrt{5}+\frac{1}{2})}{\log a} - 1.$$

Proposition 4. *Given the rank of a large Fibonacci number, to find the number of digits in it.*

Proof. Let k be the rank of a given large Fibonacci number and let the number of its digits be n.

Then

$$10^n \geq u_k \geq 10^{n-1}.$$

Also $$u_k = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right\}.$$

Since $|\tfrac{1}{2}(1-\sqrt{5} < 1)|$, it follows that

$$u_k \sim \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}.$$

Therefore given positive ε, arbitrarily small, we can find k_0 such that for $k > k_0$,

$$u_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}(1+\varepsilon).$$

Now since $u_k \geq 10^{n-1}$, we have

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}(1+\varepsilon) \geq 10^{n-1}, \text{ for } k > k_0.$$

$$\therefore \quad (k+1)\log_{10}\left(\frac{1+\sqrt{5}}{2}\right) \geq (n-1) + \tfrac{1}{2}\log_{10} 5 - \log(1+\varepsilon)$$

$$\therefore \quad n \leq (k+1)\log_{10}\left(\frac{1+\sqrt{5}}{2}\right) + 1 - \tfrac{1}{2}\log_{10} 5 - \log(1+\varepsilon).$$

Also $$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}(1+\epsilon) \text{ will be } < 10^{n}, \text{ for } k > k_0.$$

if $$n > (k+1)\log_{10}\left(\frac{1+\sqrt{5}}{2}\right) - \tfrac{1}{2}\log_{10} 5 + \log(1+\epsilon).$$

Thus n is the greatest integer in

$$(k+1)\log_{10}\left(\frac{1+\sqrt{5}}{2}\right) + 1 - \tfrac{1}{2}\log_{10} 5$$

Proposition 5. *The number of Fibonacci numbers having the same number of digits is 4 or 5.*

Proof. The Fibonacci numbers consisting of n digits satisfy the inequalities

$$10^{n} > u_k \geq 10^{n-1}.$$

Also, for $$k > k_0,\ u_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}(1+\epsilon).$$

Proceeding as in Proposition 4, it can be shown that

$$k \geq \frac{\{(n-1)+\tfrac{1}{2}\log_{10}5 + \log(1+\epsilon)\}}{\log_{10}(\tfrac{1}{2}(1+\sqrt{5}))} - 1 \text{ and } k < \frac{\{n+\tfrac{1}{2}\log_{10}5 + \log(1+\epsilon)\}}{\log_{10}(\tfrac{1}{2}(1+\sqrt{5}))} - 1.$$

The difference of the two expressions in these inequalities is $1/\log_{10}(\frac{1}{2}+\sqrt{5}/2)$ and this lies between 4 and 5 and is approximmately $= 4.787$.
It follows that k can have 4 values or 5 and hence the theorem is proved.

Note. From a table of Fibonacci numbers which can be easily constucted, it is seen that the number of Fibonacci numbers having the same number of digits is 5 or 4, these numbers recurring periodically in the order (5, 5, 5, 4) or (5, 5, 5, 5, 4).

Proposition 6. *If u_m and u_{m+1} be two consecutive Fibonacci numbers consisting of n digits and $(n+1)$ digits respectively, the number of divisions necessary to find the greatest common divisor of u_m and u_{m+1} by the usual process, is exactly $5n$ when $n \leq 3$ and is $< 5n$ when $n \geq 4$.*

Proof According to Euclid's algorithm, we have the set of equations

$$\begin{aligned}
&u_{m+1}=1.\ u_m+u_{m-1},\\
&u_m=1.\ u_{m-1}+u_{m-2},\\
&\dots\quad \dots\quad \dots\quad \dots\\
&u_3=1.\ u_2+u_1,\\
&u_2=2.\ u_1
\end{aligned}$$

Thus m divisions are necessary to find the g. c. d. of any two consecutive Fibonacci numbers u_m and u_{m+1}. But from a table of Fibonacci numbers, it is seen that m has the values 5, 10, 15 when $n=1, 2, 3$ respectively and thus $m=5n$ in these cases only and when $n=4, 6, \dots$, it follows from Proposion 5 that $m < 5n$. Hence the proposition is proved.

Corollary 1. We have

$$\begin{aligned}
&u_{m+1}+k\ u_m=(k+1)\ u_m+u_{m-1},\\
&u_m=1.\ u_{m-1}+u_{m-2},\ \text{etc.}
\end{aligned}$$

From the 2nd equation onwards, this set is identical with the set of equations shown in the proposition. Hence it follows that the number of divisions necessary to find the g. c. d. of u_m and $(u_{m+1}+k\ u_m)$ is the same as in the proposition, k being any positive integer.

Corollary 2. If u_m and u_{m+1} consist of the same number (n) of digits, it is easily seen that $m < 5n$

Corollary 3. If we have the set of equations

$$\begin{aligned}
&a=k\cdot u_m+u_{m-r},\ k \geq 1,\ r \geq 2,\\
&u_m=l.\ u_{m-r}+u_{m-r-1},\ l \geq 2,\\
&u_{m-r}=1.\ u_{m-r-1}+u_{m-r-2},\\
&\dots\quad \dots\quad \dots\quad \dots\quad \dots\quad \dots\\
&u_3=1.\ u_2+u_1,\\
&u_2=2.\ u_1,
\end{aligned}$$

where a, k, l, r are positive integers and the u's are Fibonacci numbers, it is seen that $(m-r+1)$ divisions only are required, so that the number of divisions is $< 5n$, in this case also.

It may be mentioned in this connection that Lamé, [Dickson, 1923] used the Fibonacci series to prove, by a method different from the above, that the number of divisions necessary to find the g. c. d. of two integers by the usual process of division does not exceed five times the number of digits in the smaller integer. Now the proposition 6 and corollaries show precisely that in certain cases the number of divisions required is exactly 5 times the number of digits in the smaller integer and in certain cases it is less than 5 times it.

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ON CERTAIN THEOREMS IN OPERATIONAL CALCULUS WITH *n* VARIABLES.

By

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1. The object of the present note is to obtain some theorems in Operational Calculus with n-variables represented by

$$\varphi\,(p_1, p_2,\ldots, p_n) \underset{n}{\subset} f\,(x_1, x_2,\ldots, x_n), \quad \text{or}$$

$$f(x_1, x_2, \ldots, x_n) \underset{n}{\supset} \varphi\,(p_1, p_2,\ldots, p_n),$$

to define the convergent n-ple integral

$$\frac{\varphi(p_1, p_2,\ldots, p_n)}{p_1 p_2 \cdots p_n} = \int_0^\infty \ldots \int_0^\infty \exp\,(-\Sigma\, p_i x_i) f\,(x_1, x_2,\ldots, x_n)\, dx_1 \ldots dx_n,$$

where the summation extends from $i = 1$ to $i = n$,

[*Vide* Delerue, 1951], $R\,(p_i) > 0$.

2. Theorem 1. *If* $f(p) \subset h(x)$ *and* $\varphi_s^a(p) \subset L_s^a(x) h(x)$, $L_s^a(x)$ *being the Laguerre polynomial defined by*

$$L_s^a(x) = \frac{x^{-a}}{s\,!}\, e^x \frac{d^s}{dx^s}\,(e^{-x} x^{s+a}),\ R(\alpha) > -1,\ \textit{then}$$

$$\frac{1}{m_1\,!\,m_2\,!\,..m_n\,!}\ \frac{\partial^{m_1+\cdots+m_n}}{\partial p_1^{m_1}\cdots\partial p_n^{m_n}} \left[\frac{p_1^{m_1+a_1}\ldots p_n^{m_n+a_n}\ f(p_1+\ldots+p_n)}{p_1+p_2+\ldots+p_n}\right]$$

$$= \frac{p_1^{a_1}\, p_2^{a_2}\ldots p_n^{a_n}}{p_1+p_2+\ldots+p_n} \cdot \sum_{s=0}^{m_1+\ \ +m_n} C_s\, \varphi_s^a\ (p_1+p_2+\ldots+p_n),$$

where
$$C_s = \frac{(\alpha_1+1)_{m_1}}{m_1\,!} \ldots\ldots \frac{(\alpha_n+1)_{m_n}}{m_n\,!} \times$$

$$F_A(\alpha+1\,;\,-m_1,\ldots,\,-m_n,\,-s\,;\,\alpha_1+1,\,.\,,\,\alpha_n+1\,;\,p_1,\ldots,\,p_n,\,1)$$

is the Lauricella's hypergeometric function of $n+1$ *variables.*

From $f(p) \subset h(x)$, we have

$$\frac{p_1^{m_1+a_1}\ldots p_n^{m_n+a_n}}{p_1+p_1+\ldots+p_n}\, f(p_1+p_2+\ldots+p_n)$$

$$= \int_0^\infty [e^{-p_1 x}\, p_1^{m_1+a_1}]\ldots[e^{-p_n x}\, p_n^{m_n+a_n}]\ h\,(x)\, dx.$$

So,
$$\frac{\partial^{m_1+\cdots+m_n}}{\partial p_1^{m_1}\ .\partial p_n^{m_n}} \left[\frac{p_1^{m_1+a_1}\ \ p_n^{m_n+a_n}}{p_1+\ldots+p_n}\ f(p_1+\ldots+p_n)\right]$$

$$= \int_0^\infty \frac{d^{m_1}}{d(p_1 x)^{m_1}} \left[e^{-p_1 x}(p_1 x)^{m_1-\alpha_1}\right] \dots \frac{d^{m_n}}{d(p_n x)^{m_n}} \left[e^{-p_n x}(p_n x)^{m_n+\alpha_n}\right] \frac{h(x)\,dx}{x^{\alpha_1+\dots+\alpha_n}}$$

(provided that the differentiation w. r. t. the parameters p_i under the integration sign is permissible.)

$$= [m_1!\, m_2! \dots m_n!]\, p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} \int_0^\infty \exp(-x \Sigma p_i)\, L_{m_1}^{\alpha_1}(p_1 x) \dots L_{m_n}^{\alpha_n}(p_n x)\, h(x)\, dx$$

$$= [m_1!\, m_2! \dots m_n!]\, p_1^{\alpha_1} \dots p_n^{\alpha_n} \sum_{s=0}^{m_1+\dots+m_n} C_s \int_0^\infty \exp(-x \Sigma p_i)\, L_s^{\alpha}(x)\, h(x)\, dx$$

on using the expansion $L_{m_1}^{\alpha_1}(p_1 x) \dots L_{m_n}^{\alpha_n}(p_n x) = \sum_{s=0}^{m_1+\dots+m_n} C_s L_s^{\alpha}(x)$, due to

Erdélyi, [Erdélyi, 1938, p. 154] C_s having the same definition as above.

Thus
$$\frac{1}{m_1! \dots m_n!} \frac{\partial^{m_1+\dots+m_n}}{\partial p_1^{m_1} \dots \partial p_n^{m_n}} \left[\frac{p_1^{m_1+\alpha_1} \dots p_n^{m_n+\alpha_n} f(p_1+\dots+p_n)}{p_1+\dots+p_n}\right]$$

$$= \frac{p_1^{\alpha_1}\, p_2^{\alpha_2} \dots p_n^{\alpha_n}}{p_1+\dots+p_n} \sum_{s=0}^{m_1+\dots+m_n} C_s \varphi_s^{\alpha}(p_1+\dots+p_n)$$

which establishes the theorem.

As regards the differentiation w. r. t. the parameters p_i under the sign of integration effected above we see that all the conditions to be satisfied for such operation (*vide* Carslaw, 1930, p. 200) hold, if only we assume that

$$h(x) = O(x^{\mu}), \qquad \text{for small } x, \quad R(\mu) > 0$$
$$= O\{\exp(-x^{\lambda})\}, \quad \text{for large } x, \quad R(\lambda) > 0.$$

As an application, let $h(x) = e^{-x} x^{\alpha} L_n^{\alpha}(x)$;

so that
$$f(p) = \frac{\Gamma(n+\alpha+1)}{n!} \frac{p^{n+1}}{(1+p)^{n+\alpha+1}}$$

and
$$L_s^{\alpha}(x)\, h(x) = e^{-x} x^{\alpha} L_n^{\alpha}(x)\, L_s^{\alpha}(x)$$

So
$$\varphi_s^{\alpha}(p) = \frac{\Gamma(n+\alpha+1)\,\Gamma(s+\alpha+1)}{n!\, s!\, \Gamma(\alpha+1)} \frac{(p-1)^{n+s+\alpha+1}}{p^{n+s+2\alpha+1}}\, {}_2F_1\left(\begin{matrix} n+\alpha+1, & s+\alpha+1 \\ & \alpha+1 \end{matrix}; \frac{1}{p^2}\right);$$

by Howell [Howell, 1937, p. 391, 403].

Then, from the theorem, we obtain

$$\frac{\Gamma(\alpha+1)}{m_1!\, m_2! \dots m_n!} \cdot \frac{\partial^{m_1+\dots+m_n}}{\partial p_1^{m_1} \dots \partial p_n^{m_n}} \left[\frac{p_1^{m_1+\alpha_1} \dots p_n^{m_n+\alpha_n} (p_1+\dots+p_n)^n}{(1+p_1+\dots+p_n)^{n+\alpha+1}}\right]$$

$$= \frac{p_1^{\alpha_1} \dots p_n^{\alpha_n} (p_1+\dots+p_n-1)}{(p_1+\dots+p_n)^{n+2\alpha+2}} \cdot \sum_{s=0}^{m_1+\dots+m_n} \left(1 - \frac{1}{p_1+\dots+p_n}\right)^s \frac{\Gamma(s+\alpha+1)}{s!} C_s \times$$
$$\times\, {}_2F_1\left[\begin{matrix} n+\alpha+1, & s+\alpha+1 \\ & \alpha+1 \end{matrix}; \frac{1}{(p_1+\dots+p_n)^2}\right].$$

3. **Theorem II** *If* $h(p) \subset g(x)$ *and*

$$\varphi(p_1, \ldots, p_n) \subset x_1^{\beta_1 - 1} \ldots x_n^{\beta_n - 1} (1+x_1)^{-\alpha_1} \ldots (1+x_n)^{-\alpha_n} \frac{h(x_1 + \ldots + x_n)}{x_1 + \ldots + x_n},$$

then $$[\Gamma(\alpha_1) \ldots \Gamma(\alpha_n)] \frac{\varphi(p_1, \ldots, p_n)}{p_1 p_2 \ldots p_n}$$

$$= \int_0^\infty \frac{E(\alpha_1, \beta_1 :: p_1 + s) \ldots E(\alpha_n, \beta_n :: p_n + s)}{(p_1+s)^{\beta_1} \ldots (p_n+s)^{\beta_n}} g(s)\, ds; \quad E(\alpha, \beta :: x)$$

being McRobert's E-functian defined by

$$E(\alpha, \beta :: x) = \Gamma(\alpha) \int_0^\infty e^{-\lambda} \lambda^{\beta - 1} \left(1 + \frac{\lambda}{x}\right)^{-\alpha} d\lambda, \qquad R(\beta) > 0, \ |\,\mathrm{am}\, x\,| < \pi.$$

We have $$\frac{\varphi(p_1, \ldots, p_n)}{p_1 p_2 \ldots p_n} = \int_0^\infty \ldots \int_0^\infty \exp(-x \Sigma p_i) \ [x_1^{\beta_1 - 1} \ldots x_n^{\beta_n - 1}][(1+x_1)^{-\alpha_1} \ldots (1+x_n)^{-\alpha_n}]$$

$$\times \frac{h(x_1 + \ldots + x_n)}{x_1 + \ldots + x_n} dx_1 \ldots dx_n \tag{3.1}$$

and $$\frac{h(x_1 + \ldots + x_n)}{x_1 + \ldots + x_n} = \int_0^\infty \exp(-s \Sigma x_i). \ g(s)\, ds. \tag{3.2}$$

Substituting (3.2) in the R. H. S. of (3.1) and then changing the order of integration, we obtain

$$\frac{\varphi(p_1, \ldots, p_n)}{p_1 p_2 \ldots p_n} = \int_0^\infty g(s)\, ds \int_0^\infty e^{-x_1(s+p_1)} x_1^{\beta_1 - 1} (1+x_1)^{-\alpha_1} dx_1 \int_0^\infty \ldots \int_0^\infty e^{-x_n(s+p_n)} x_n^{\beta_n - 1} (1+x_n)^{-\alpha_n} dx_n$$

$$= \frac{1}{\Gamma(\alpha_1) \ldots \Gamma(\alpha_n)} \int_0^\infty \frac{E(\alpha_1, \beta_1 :: p_1 + s) \ldots E(\alpha_n, \beta_n :: p_n + s)}{(p_1+s)^{\beta_1} \ldots (p_n+s)^{\beta_n}} g(s)\, ds,$$

which proves the theorem.

As regards the change in the order of integration effected above we can proceed as in (Chakravarty, 1953*a*). For this we can assume

$$g(x) = O(x^{\mu}), \qquad \text{for small } x, \quad R(\mu) > 0$$
$$= O\{\exp(-x^{\lambda})\}, \quad \text{for large } x, \quad R(\lambda) > 0$$

and integral (3.1) is absolutely convergent.

As an example, we can take $h(x) = g(x) = 1$, so that

$$\varphi(p) = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{p^{\beta - 2}} E(\alpha, \beta - 1 :: p) \subset x^{\beta - 2} (1+x)^{-\alpha};$$

we then obtain the integral

$$\int_0^\infty \frac{E[\alpha, \beta :: \kappa(t+1)]\, dt}{(t+1)^{\beta}} = E(\alpha, \beta - 1 :: \kappa), R(\beta) > 1, \text{ obtained by}$$

Bose [Bose, 1951, *pp.* 160–167].

5. Theorem IV *If* $\varphi(u_1, u_2, \ldots\ldots, u_n) \underset{n}{\subset} f(x_1, x_2, \ldots\ldots, x_n)$ *and* $g(x_1, x_2, \ldots\ldots, x_p) \underset{p}{\supset} \psi(u_1, u_2, \ldots\ldots, u_p)$, *then*

$$\underset{\xrightarrow[p\text{-ple. integral}]{}}{\int_0^\infty \int_0^\infty} \frac{\varphi(u_1, \ldots, u_n)\, g(u_1, \ldots, u_p)}{u_1 u_2 \ldots u_n}\, du_1 \ldots du_p$$

$$= \underset{\xrightarrow[n\text{-ple. integral}]{}}{\int_0^\infty \ldots \int_0^\infty} \frac{f(x_1, \ldots, x_n)\, \psi(x_1, \ldots, x_p)}{x_1 x_2 \ldots x_p} \exp\{-\sum_{r=p+1}^{n} u_r x_r\} dx_1 \ldots dx_n,$$ *the integrals involved being assumed convergent.*

By definition

$$\frac{\varphi(u_1, \ldots, u_n)}{u_1 u_2 \ldots u_n} = \int_0^\infty \ldots \int_0^\infty \exp(-\Sigma u_i x_i)\, f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \tag{5.1}$$

and

$$\frac{\psi(u_1, \ldots, u_p)}{u_1 u_2 \ldots u_p} = \int_0^\infty \ldots \int_0^\infty \exp(\Sigma u_i x_i)\, g(x_1, \ldots, x_p) dx_1 \ldots dx_p. \tag{5.2}$$

so that

$$\underset{\xrightarrow[p\text{-ple. integral}]{}}{\int_0^\infty \ldots \int_0^\infty} \frac{\varphi(u_1, \ldots, u_n)\, g(u_1, \ldots, u_p)}{u_1 u_2 \ldots u_n} du_1 \ldots du_p$$

$$= \underset{\xrightarrow[p\text{-ple. integral}]{}}{\int_0^\infty \ldots \int_0^\infty} \Big[g(u_1, \ldots, u_p) \underset{\xrightarrow[n\text{-ple. integral}]{}}{\int_0^\infty \ldots \int_0^\infty} \exp(-\Sigma u_i x_i)\, f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \Big] du_1 \ldots du_p$$

$$= \underset{\xrightarrow[n\text{-ple. integral}]{}}{\int_0^\infty \ldots \int_0^\infty} \exp\Big(-\sum_{r=p+1}^{n} u_r x_r\Big) f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \underset{\xrightarrow[p\text{-ple. integral}]{}}{\int_0^\infty \ldots \int_0^\infty} \exp(-\Sigma u_i x_i) \times$$

$$\times\, g(u_1, \ldots, u_p)\, du_1 \ldots du_p,$$

on changing the order of integration which is assumed legitimate by assuming

$$f(x_1, \ldots, x_n) = O(x_i^{\mu_i}) \quad \text{for small } x_i, \quad R(\mu_i) > 0, \text{ for every } i \quad (i = 1, 2, \ldots, n),$$
$$= O\{\exp(-x_i^{\lambda_i})\} \text{ for large } x_i, \quad R(\lambda_i) > 0.$$

and

$$g(x_1, \ldots, x_p) = O(x_i^{\nu_i}), \quad \text{for small } x_i, \quad R(\nu_i) > 0, \quad i = 1, 2, \ldots, p.$$
$$= O\{\exp(-x_i^{\theta_i})\}, \text{ for large } x_i, \quad R(\theta_i) > 0.$$

The theorem now follows on utilisation of the relation (5.2).

Two consequences of this theorem are worth notice

(1) when $n = p$ we obtain

$$\int_0^\infty \cdots \int_0^\infty \frac{\varphi(u_1,\ldots,u_n)\, g(u_1,\ldots,u_n)}{u_1 u_2 \ldots u_n} du_1 \ldots du_n$$

$$= \int_0^\infty \cdots \int_0^\infty \frac{f(x_1,\ldots,x_n)\, \psi(x_1,\ldots,x_n)}{x_1 x_2 \ldots x_n} dx_1 \ldots dx_n, \text{ when } \varphi(u_1,\ldots,u_n) \underset{n}{\subset} f(x_1,\ldots,x_n)$$

and $g(x_1,\ldots x_n) \underset{n}{\supset} \psi(u_1,\ldots,u_n)$. this is the analogue of Goldstein's Theorem in operational Calculus with n variables

(2) when $n = 2$, and $p = 1$, we obtain the theorem of Chakravarty obtained in a different paper [Chakravarty, 1953b]

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